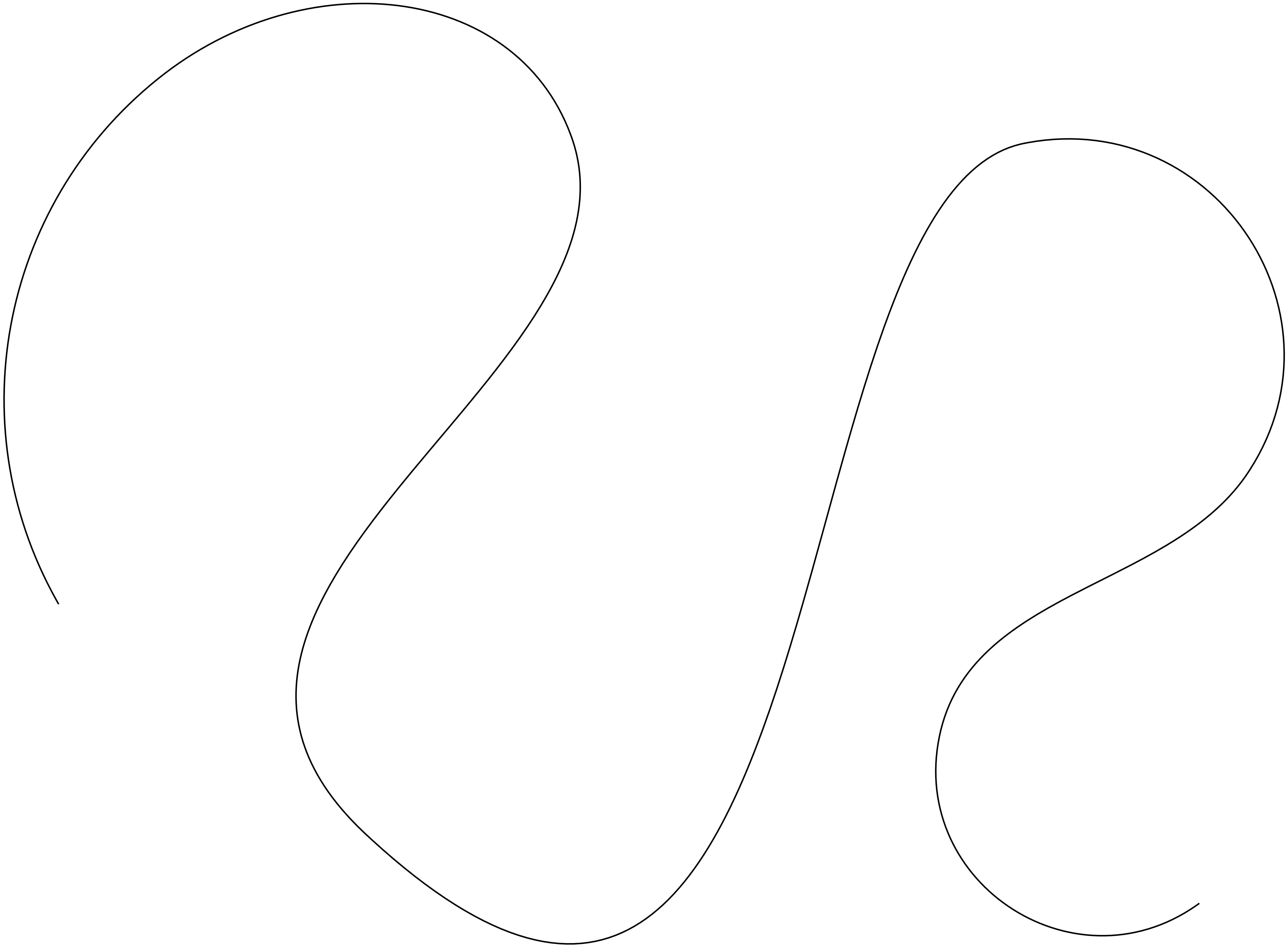
Parametric Curves

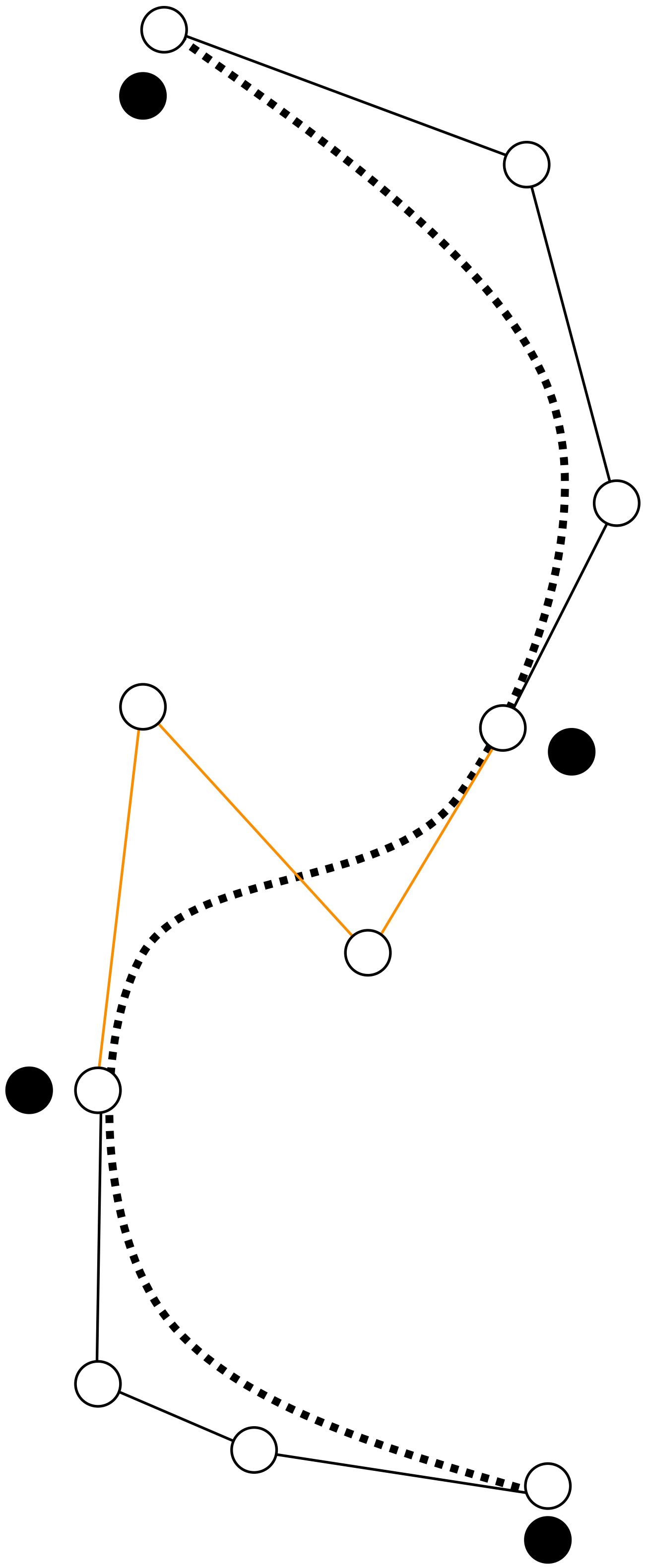
Connelly Barnes

CS4810: Introduction to Graphics

Acknowledgment: slides by Jason Lawrence, Misha Kazhdan, Allison Klein, Tom Funkhouser, Adam Finkelstein and David Dobkin

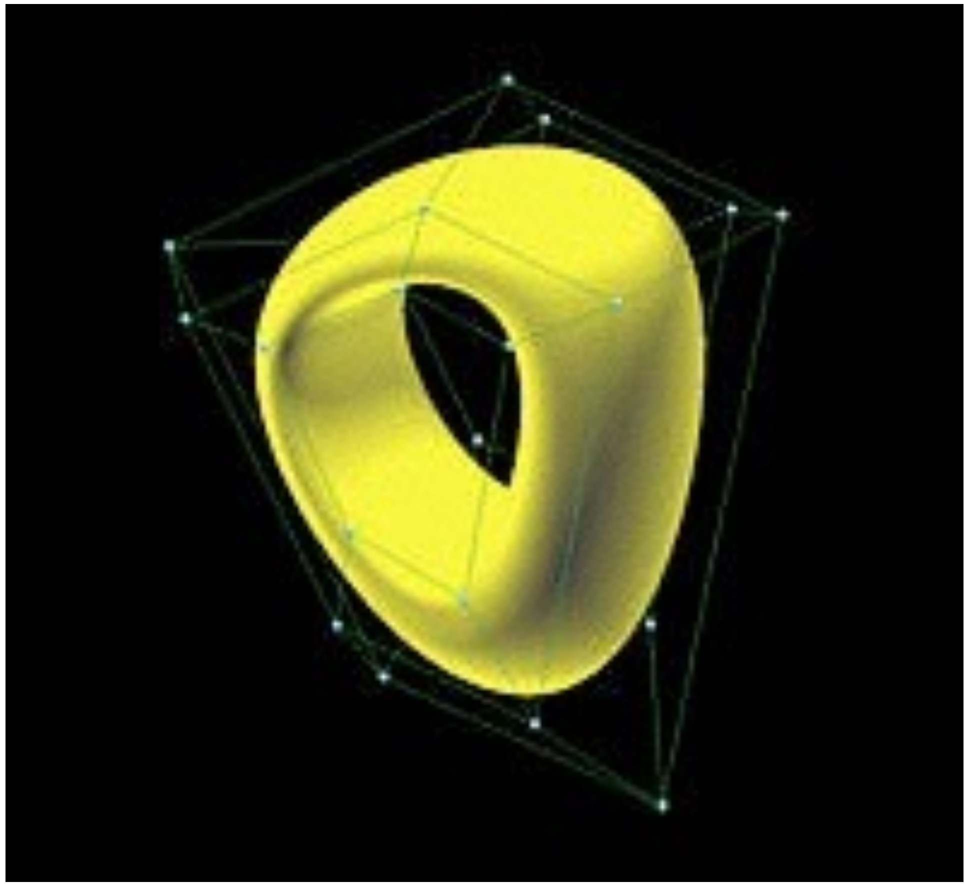


Parametric Curves and Surfaces



Part 1: Curves

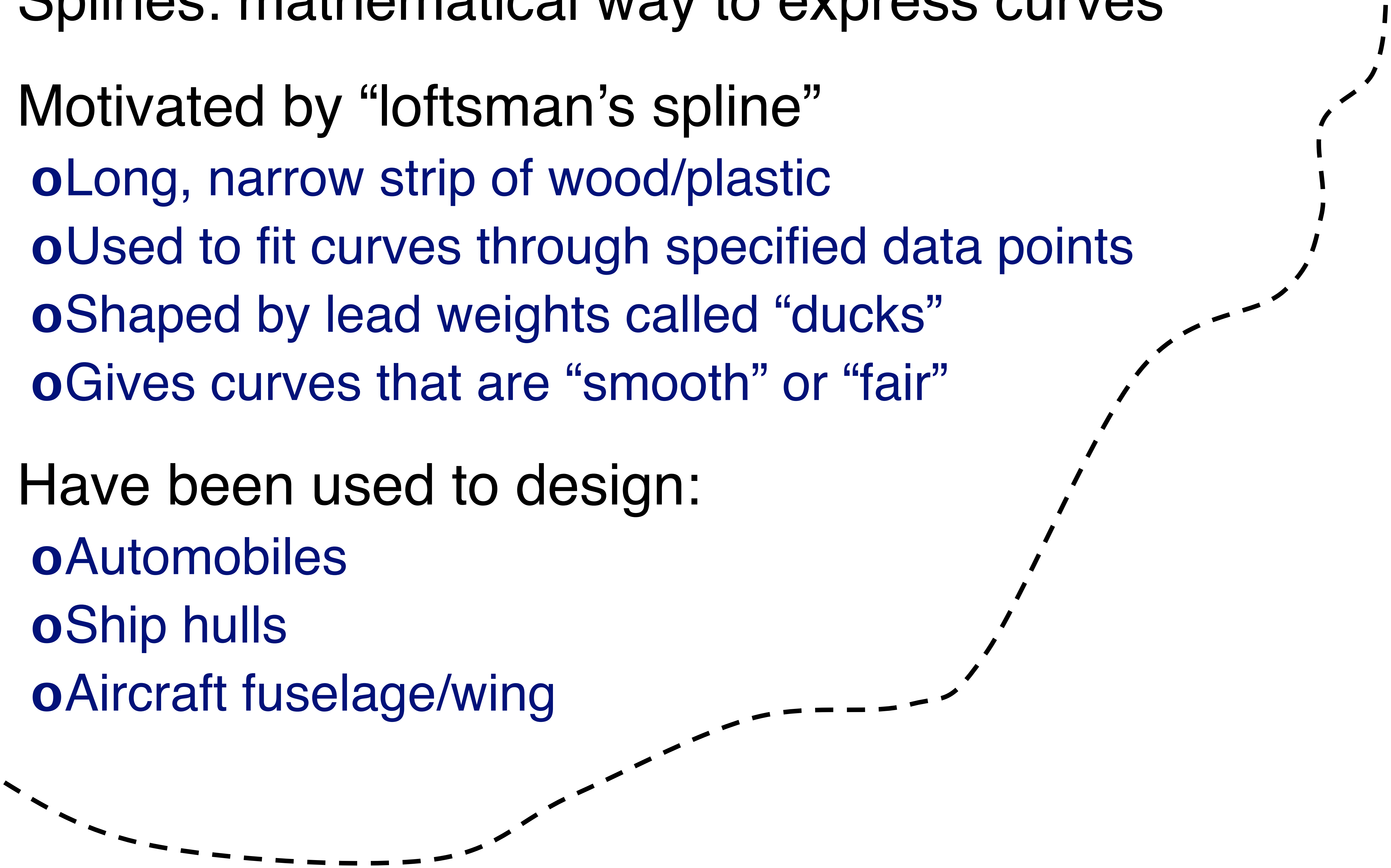
Part 2: Surfaces



Courtesy of C.K. Shene

Curves

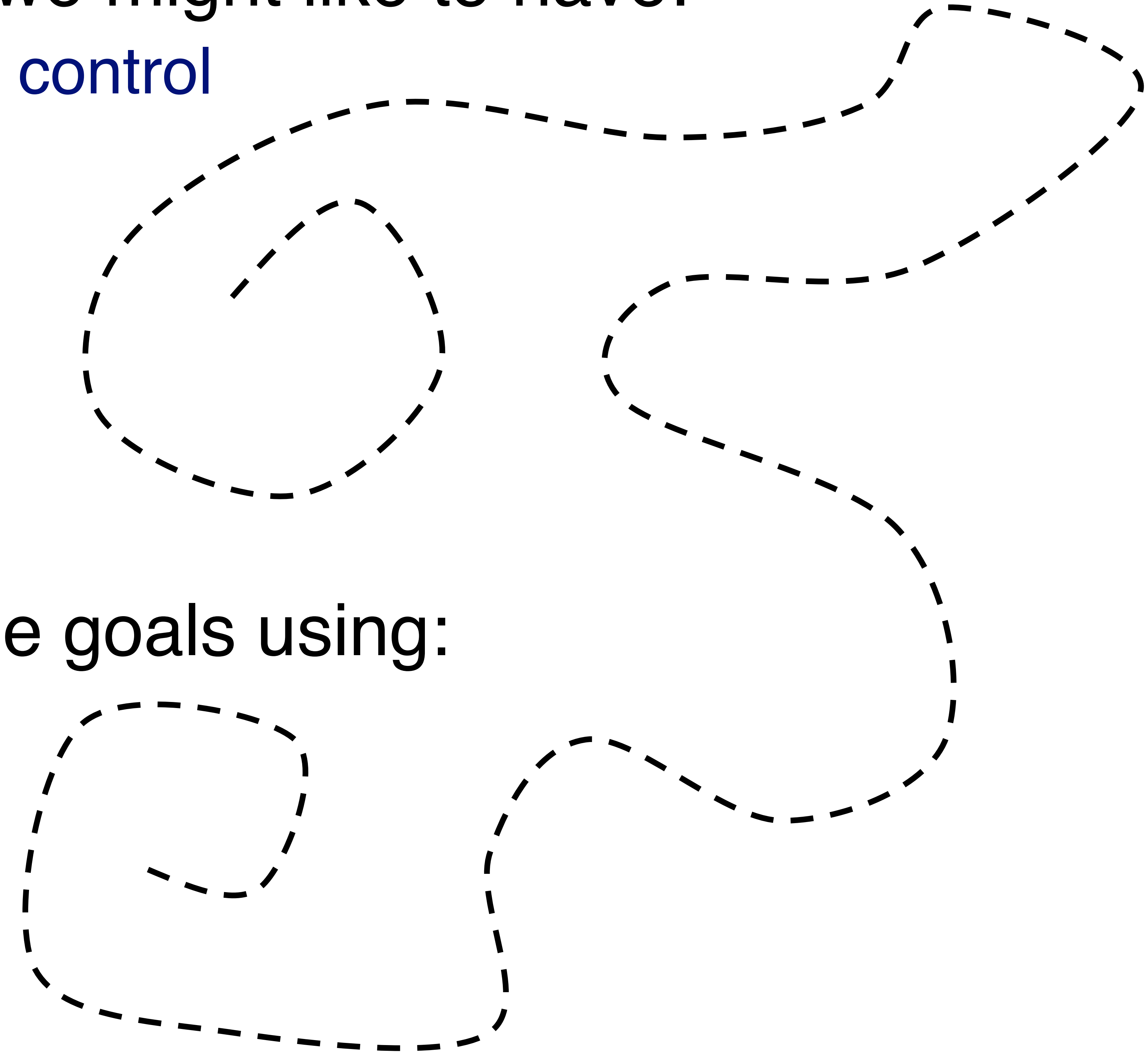
- Splines: mathematical way to express curves
- Motivated by “loftsman’s spline”
  - Long, narrow strip of wood/plastic
  - Used to fit curves through specified data points
  - Shaped by lead weights called “ducks”
  - Gives curves that are “smooth” or “fair”
- Have been used to design:
  - Automobiles
  - Ship hulls
  - Aircraft fuselage/wing



Goals

- Some attributes we might like to have:
  - Predictable/local control
  - Simple
  - Continuous

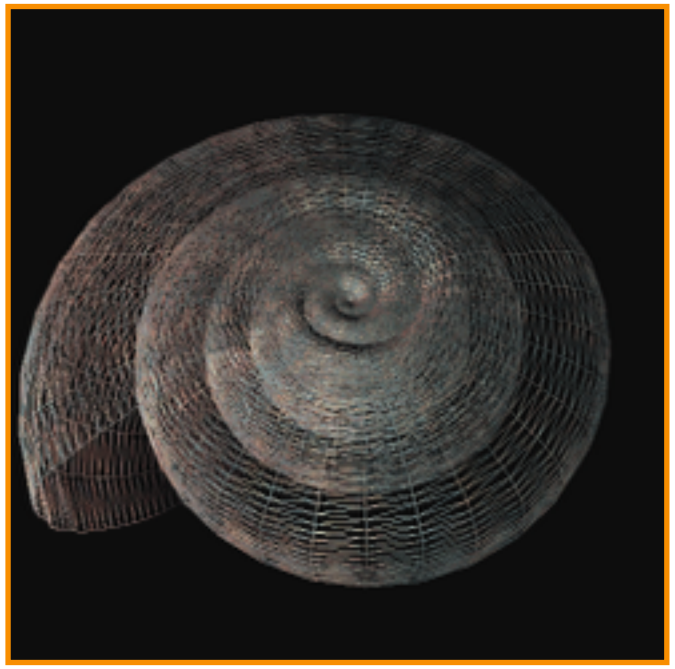
- We’ll satisfy these goals using:
  - Piecewise
  - Polynomials



Many applications in graphics

- Animation paths
- Shape modeling
- etc…

Animation
(Angel, Plate 1)

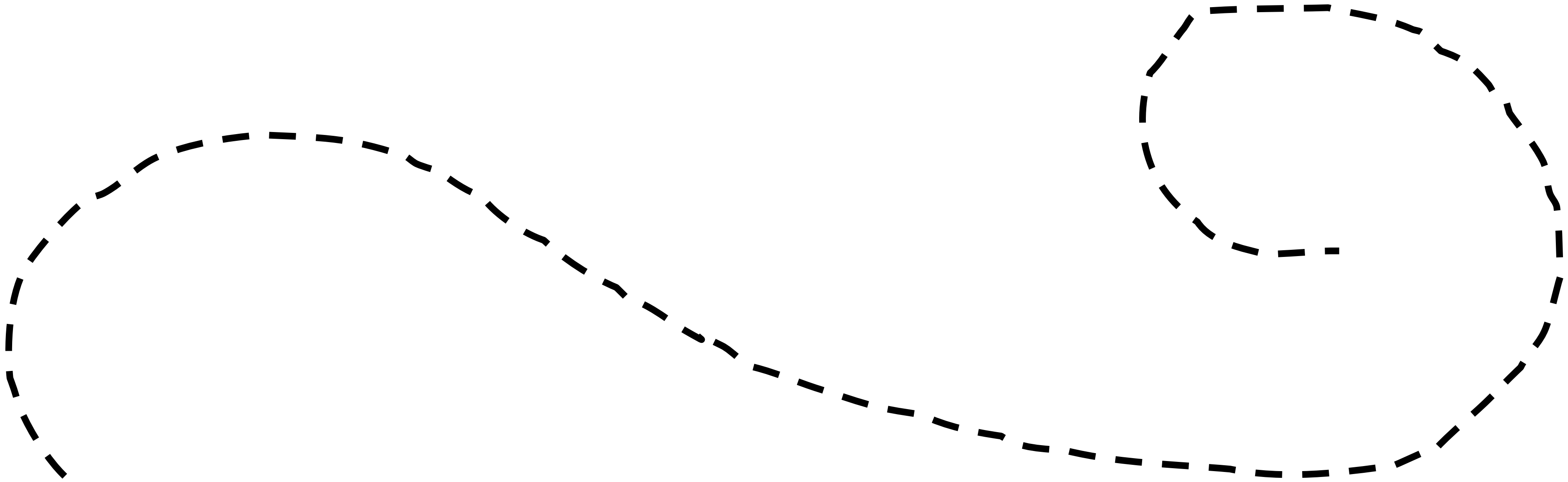


Shell
(Douglas Turnbull)

What is a Spline in CG?

A spline is a *piecewise* *polynomial function* whose derivatives satisfy some *continuity constraints* across curve boundaries.

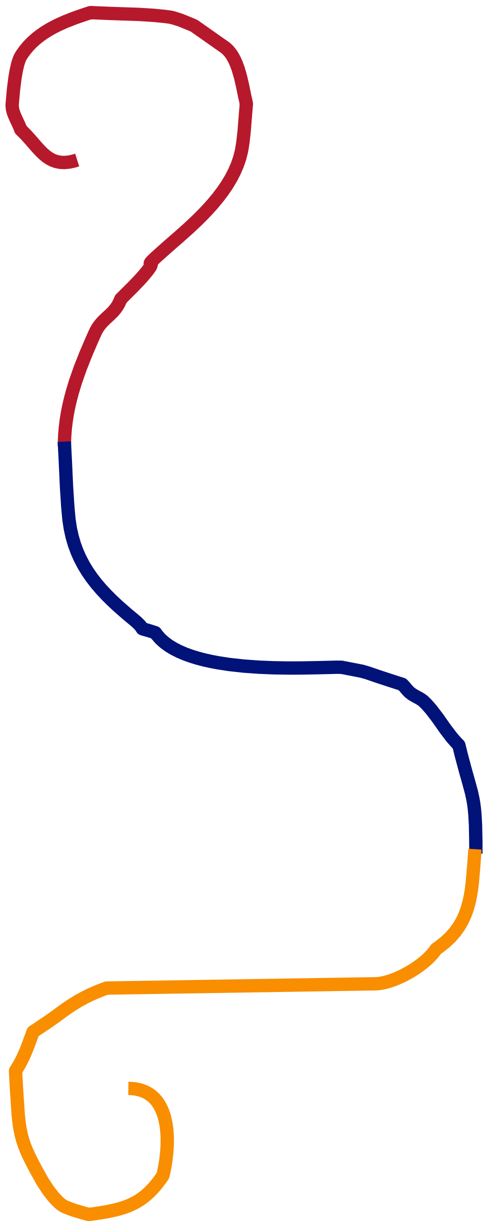
So let's look at what this means…



What is a Spline in CG?

Piecewise: the spline is actually a collection of individual segments joined together.

Polynomial functions: each of these segments is expressed by a polynomial function.

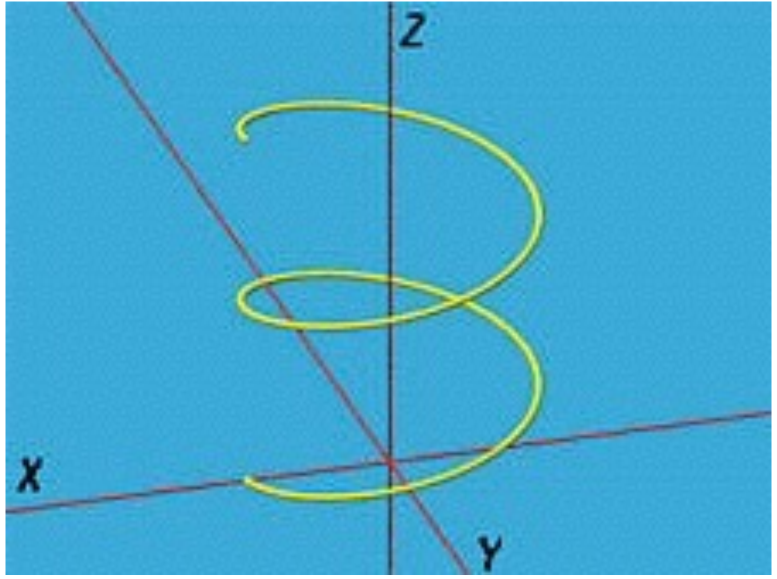


Parametric Curves

A parametric curve in *d*-dimensions is defined by a collection of 1D functions of one variable that give the coordinates of points on the curve at each value of *u*:

$$\Phi(u) = \left(x_1(u), \ldots, x_d(u)\right)$$

$$\Phi(u) = \left(\cos(u), u, \sin(u)\right)$$



Courtesy of C.K. Shene

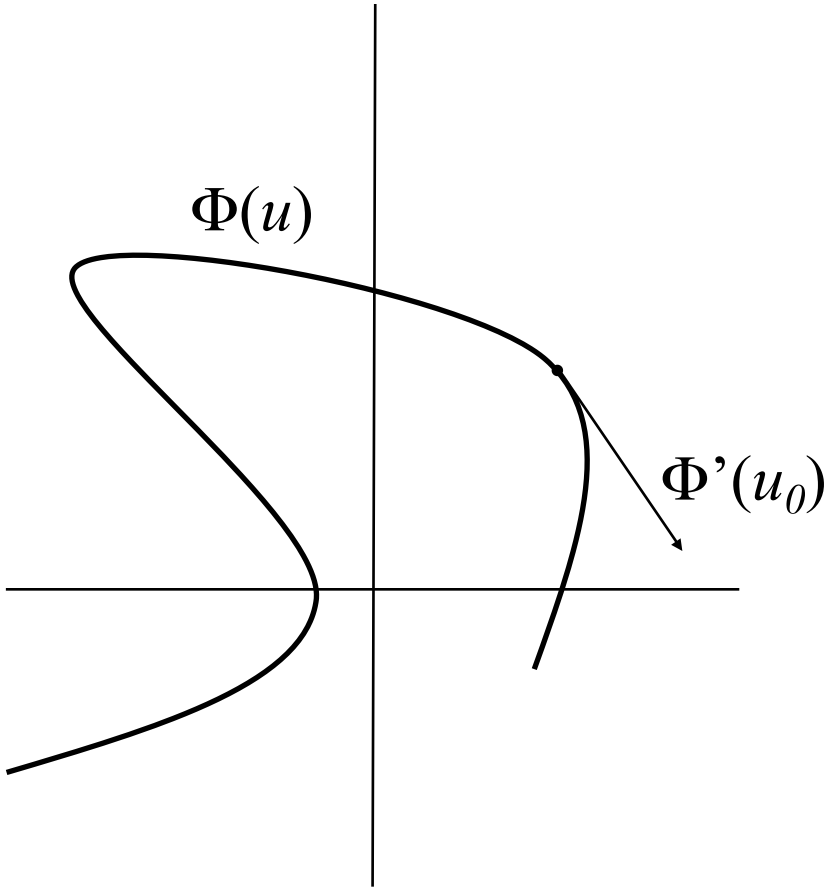
Note:
A parametric curve is **not** the graph of a function, it is the path traced out as the value of *t* is allowed to change.

Derivatives

If $\Phi(u)=(x(u),y(u))$ is the parametric equation of a curve, the parametric derivative of the curve at a point u_0 is the vector:

$$\Phi'(u_0) = \left(x'(u_0), y'(u_0)\right)$$

which points in a direction tangent to the curve.



Note:
The direction of the derivative is determined by the path that the curve traces out.

The magnitude of the parametric derivative is determined by the tracing speed.

Polynomials

A polynomial in the variable *u* is:

- “An algebraic expression written as a sum of constants multiplied by different powers of a variable.”

$$P(u) = a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n = \sum_{k=0}^{n} a_k u^k$$

The constant a_k is referred to as the <u>*k*-th coefficient</u> of the polynomial *P*.

Polynomials (Degree)

$$P(u) = a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n = \sum_{k=0}^{n} a_k u^k$$

A polynomial *P* has <u>degree</u> *n* if when written in canonical form above, the highest exponent is *n* (and a_n is nonzero).

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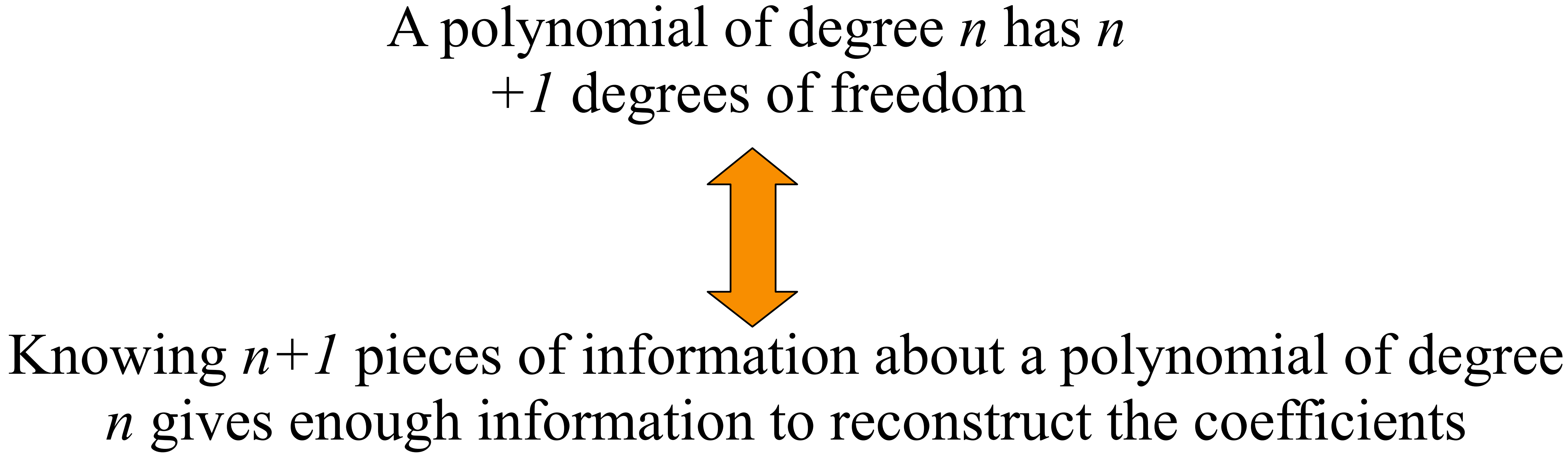


A polynomial of degree *n* has *n +1* degrees of freedom

Knowing *n+1* pieces of information about a polynomial of degree *n* gives enough information to reconstruct the coefficients

Polynomials (Matrices)

$$P(u) = a_0 + a_1 u + a_2 u^2 + \ldots + a_n u^n = \sum_{k=0}^{n} a_k u^k$$

The polynomial P can be expressed as the matrix multiplication of a column vector and a row vector:

$$P(u) = \begin{pmatrix} u^n & \cdots u^0 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix}$$

Polynomials (Matrices)

$$P(u)=\sum_{k=0}^{n} a_k u^k$$

Example:

If we know the values of the polynomial *P* at *n+1* different values:

$$P(u_0)=p_0,\ldots,P(u_n)=p_n$$

We can compute the coefficients of *P* by inverting the appropriate matrix:

$$\begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} (u_0)^n & \cdots & (u_0)^0 \\ \vdots & \ddots & \vdots \\ (u_n)^n & \cdots & (u_n)^0 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} \Longrightarrow \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} (u_0)^n & \cdots & (u_0)^0 \\ \vdots & \ddots & \vdots \\ (u_n)^n & \cdots & (u_n)^0 \end{pmatrix}^{-1} \begin{pmatrix} p_0 \\ \vdots \\ p_n \end{pmatrix}$$

Polynomials (Matrices)

$$P(u)=\sum_{k=0}^{n}a_k u^k$$

Example:

So, if we are given the values of the polynomial *P* at the *n+1* positions $u_0,\ldots,u_n$, we can compute the value of *P* at any position *u* by solving:

$$P(u)=\begin{pmatrix}u^n & \cdots & u^0\end{pmatrix}\begin{pmatrix}(u_0)^n & \cdots & (u_0)^0\\ \vdots & \ddots & \vdots\\ (u_n)^n & \cdots & (u_n)^0\end{pmatrix}^{-1}\begin{pmatrix}p_0\\ \vdots\\ p_n\end{pmatrix}$$

Parametric Polynomial Curves

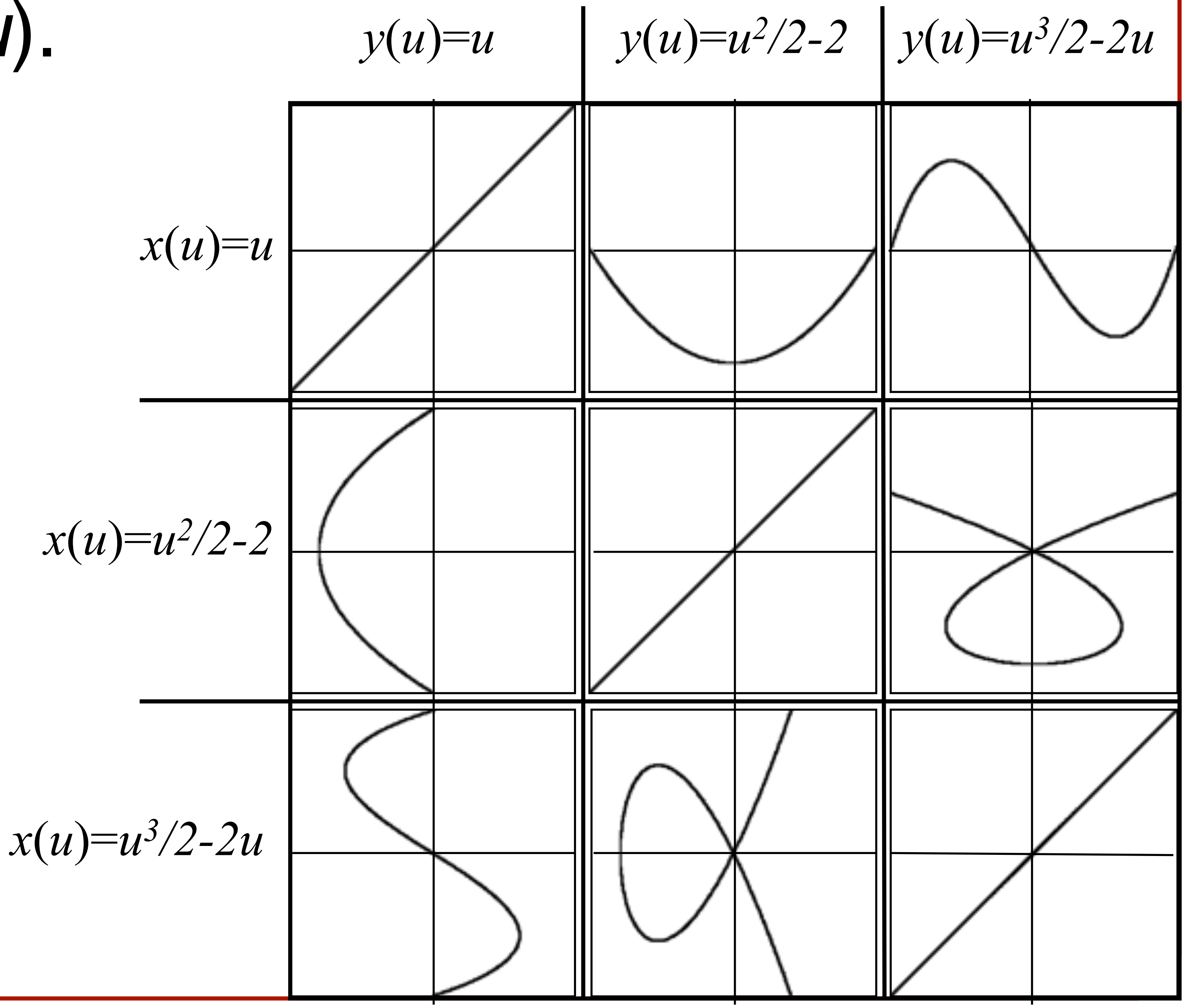
- A parametric polynomial curve of degree *n* in *d* dimensions is a collection of *d* polynomials, each of which is of degree no larger than *n*:

$$\Phi(u) = \left(x_1(u) = \sum_{k=0}^{n} a_{1,k} u^k, \ldots, x_d(u) = \sum_{k=0}^{n} a_{d,k} u^k \right)$$

Parametric Polynomial Curves

Examples:

- When $x(u)=u$, the curve is just the graph of $y(u)$.
- Different parametric equations can trace out the same curve.
- As the degree gets larger, the complexity of the curve increases.



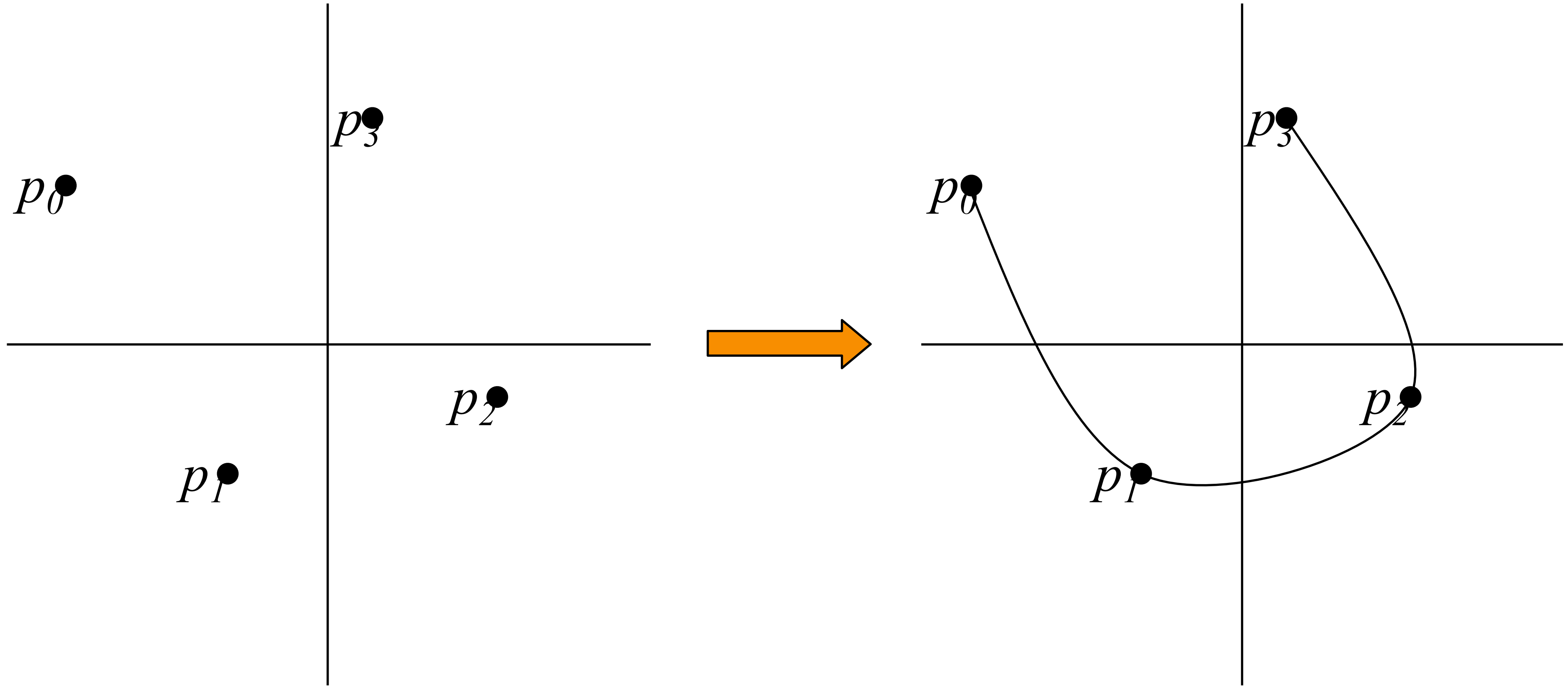
Parametric Curves

Goal:

Given a collection of *m* points in *d* dimensions:

$$\{p_1 = (x_{1,1},\ldots,x_{1,d}),\ldots,p_m = (x_{m,1},\ldots,x_{m,d})\}$$

define a parametric curve that passes through (or near) the points



Parametric Curves

Direct Approach:

Solve for the *m* coefficients of a parametric polynomial curve of degree *m-1*, passing through the points.

Parametric Curves

Direct Approach:

Solve for the *m* coefficients of a parametric polynomial curve of degree *m-1*, passing through the points.

Limitations:

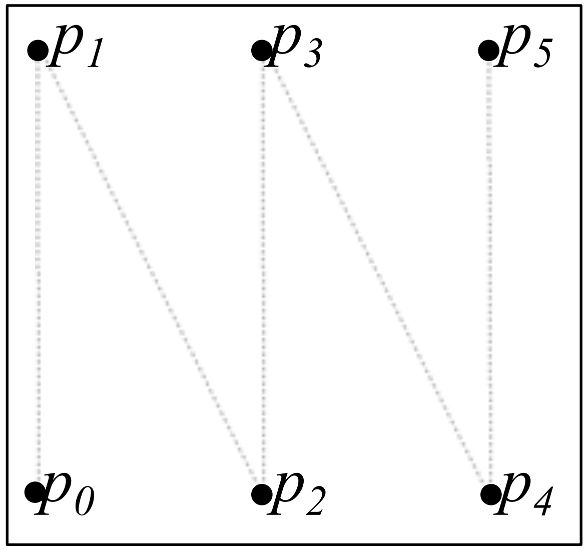
- No local control
- As the number of points increases, the dimension gets larger, and the curve oscillates more.

Splines

Approach:

Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- Local Control
- Simplicity
- Continuity/Smoothness



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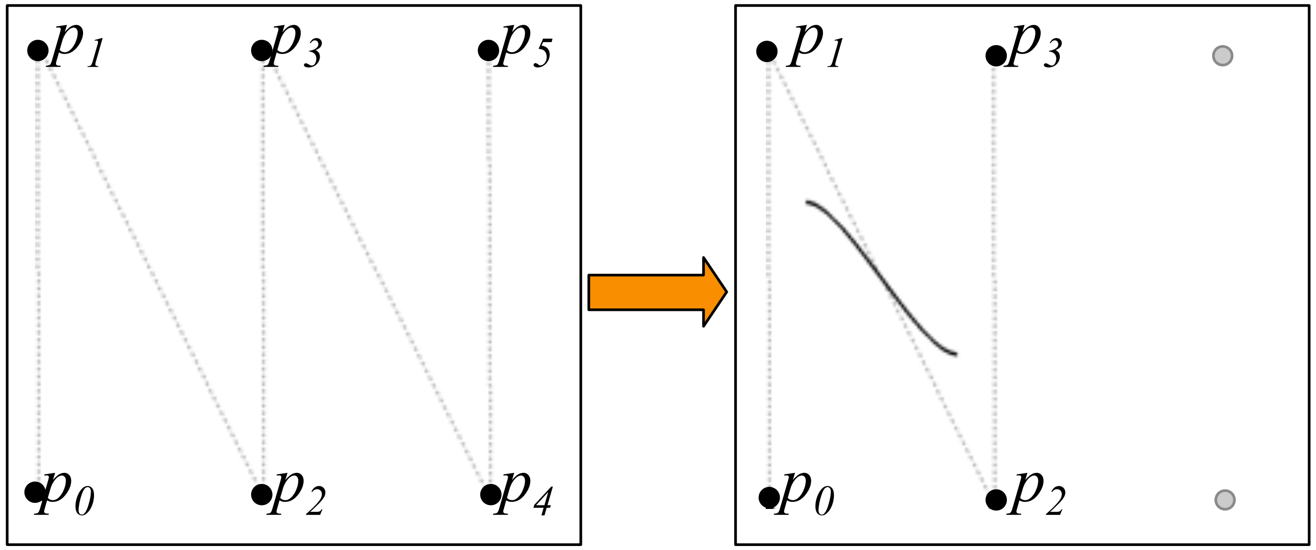

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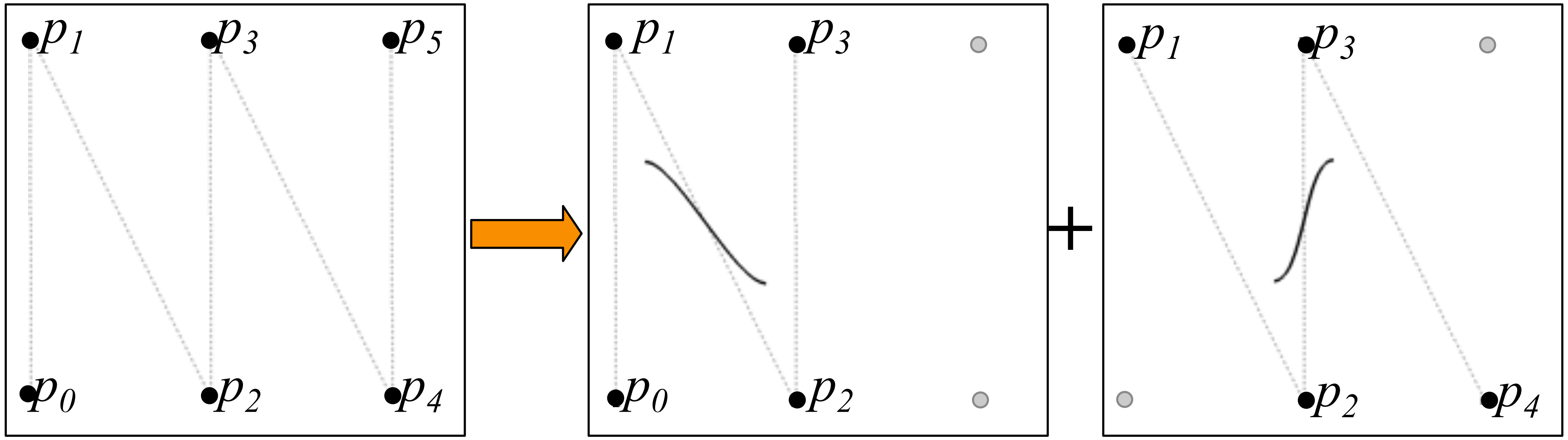

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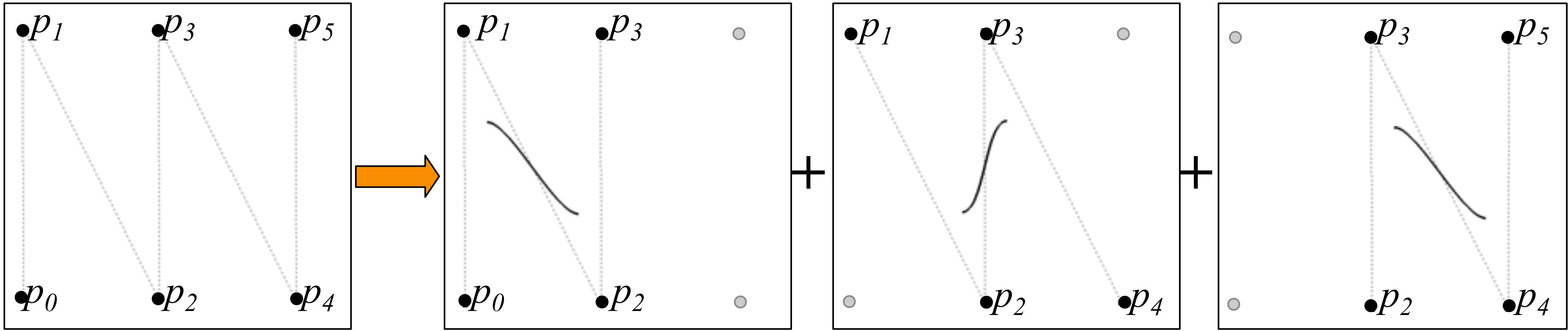

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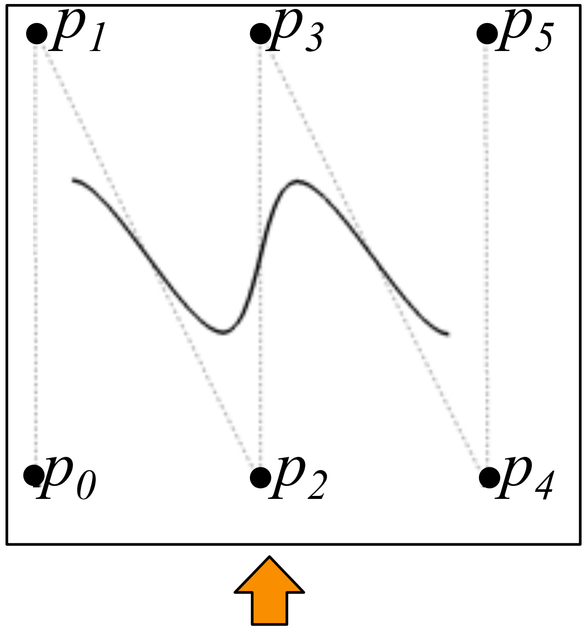

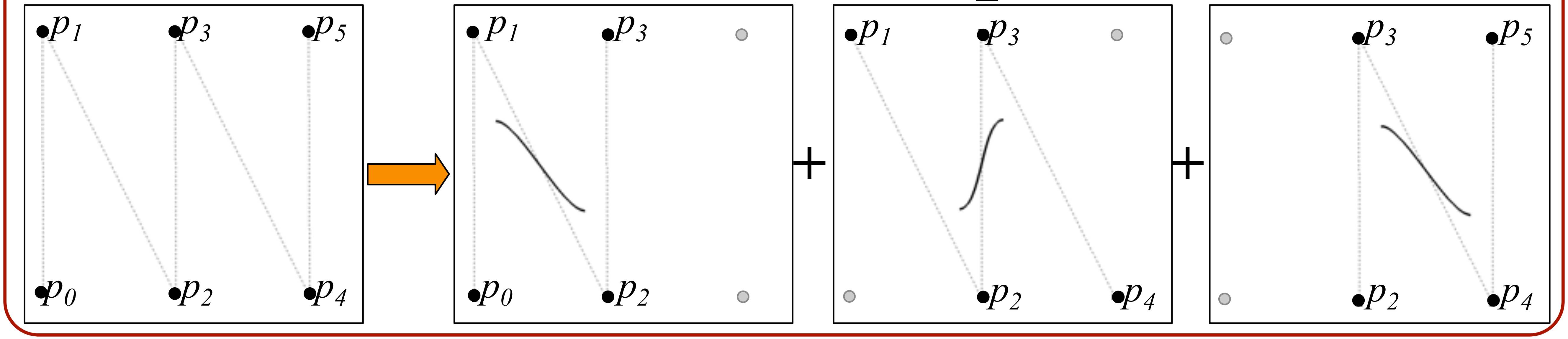

Piecewise parametric polynomials

Approach:

Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- Local Control:
  - Individual curve segments are defined using only local information
- Simplicity
  - Curve segments are low-order polynomials

Piecewise parametric polynomials

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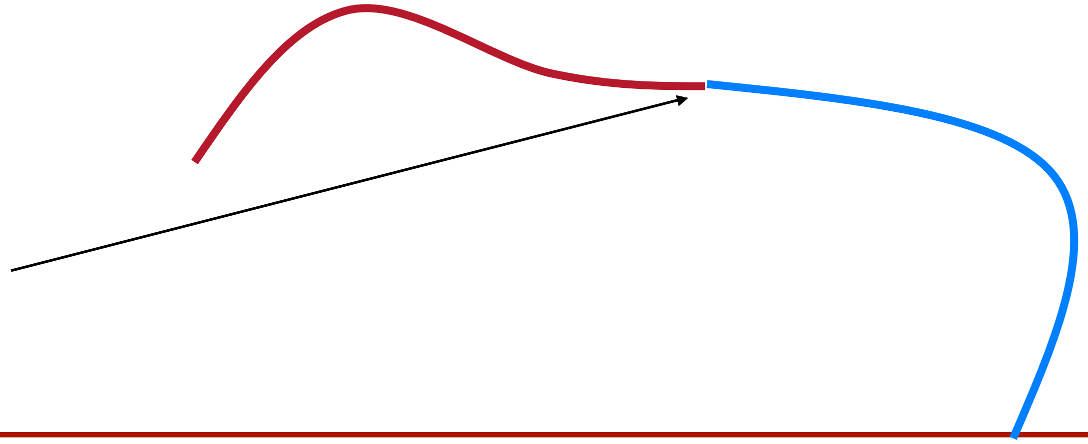
Fit low-order polynomials to groups of points so that the combined curve passes through (or near) the points while providing:

- **o**Local Control:
  - »Individual curve segments are defined using only local information
- **o**Simplicity
  - »Curve segments are low-order polynomials
- **o**Continuity/Smoothness
  - »How do we guarantee smoothness at the joints?

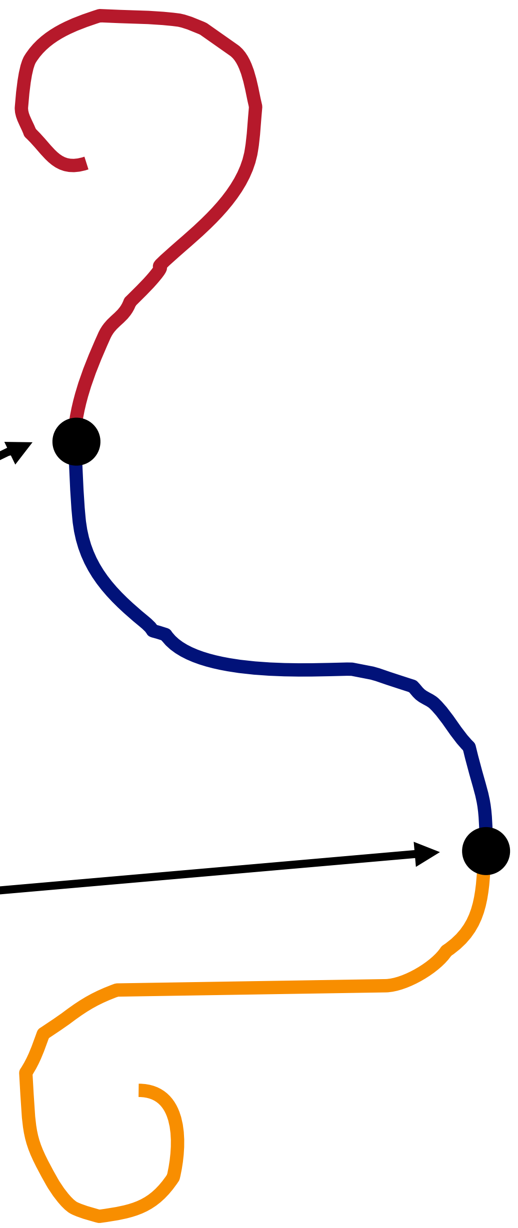


Continuity/Smoothness

Continuity:

Within the parameterized domain, the polynomial functions are continuous and smooth.

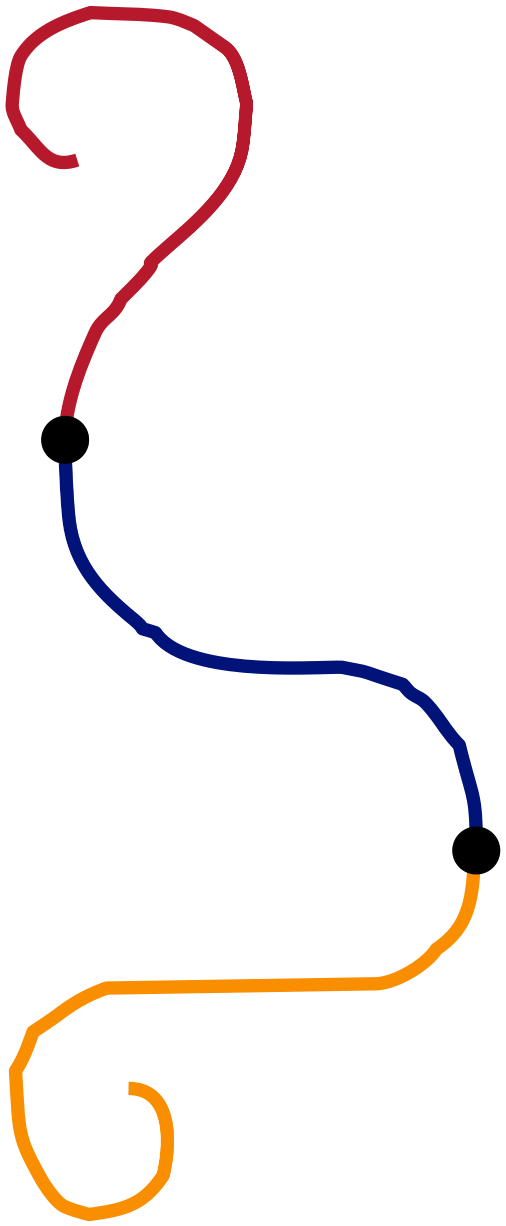
The derivatives of our polynomial functions must satisfy continuity constraints across the curve boundaries.



Continuity/Smoothness

Parametric continuity: derivatives of the two curves are *equal* where they meet.

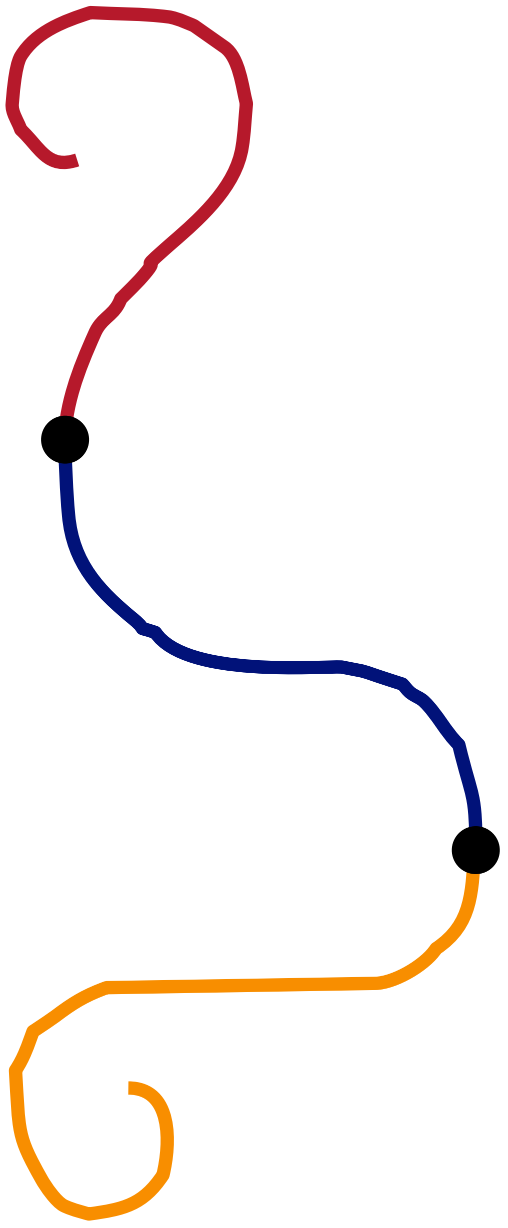
- C^0 means two curves just meet
- C^1 means 1st derivatives equal
- C^2 means both 1st and 2nd derivates equal



Continuity/Smoothness

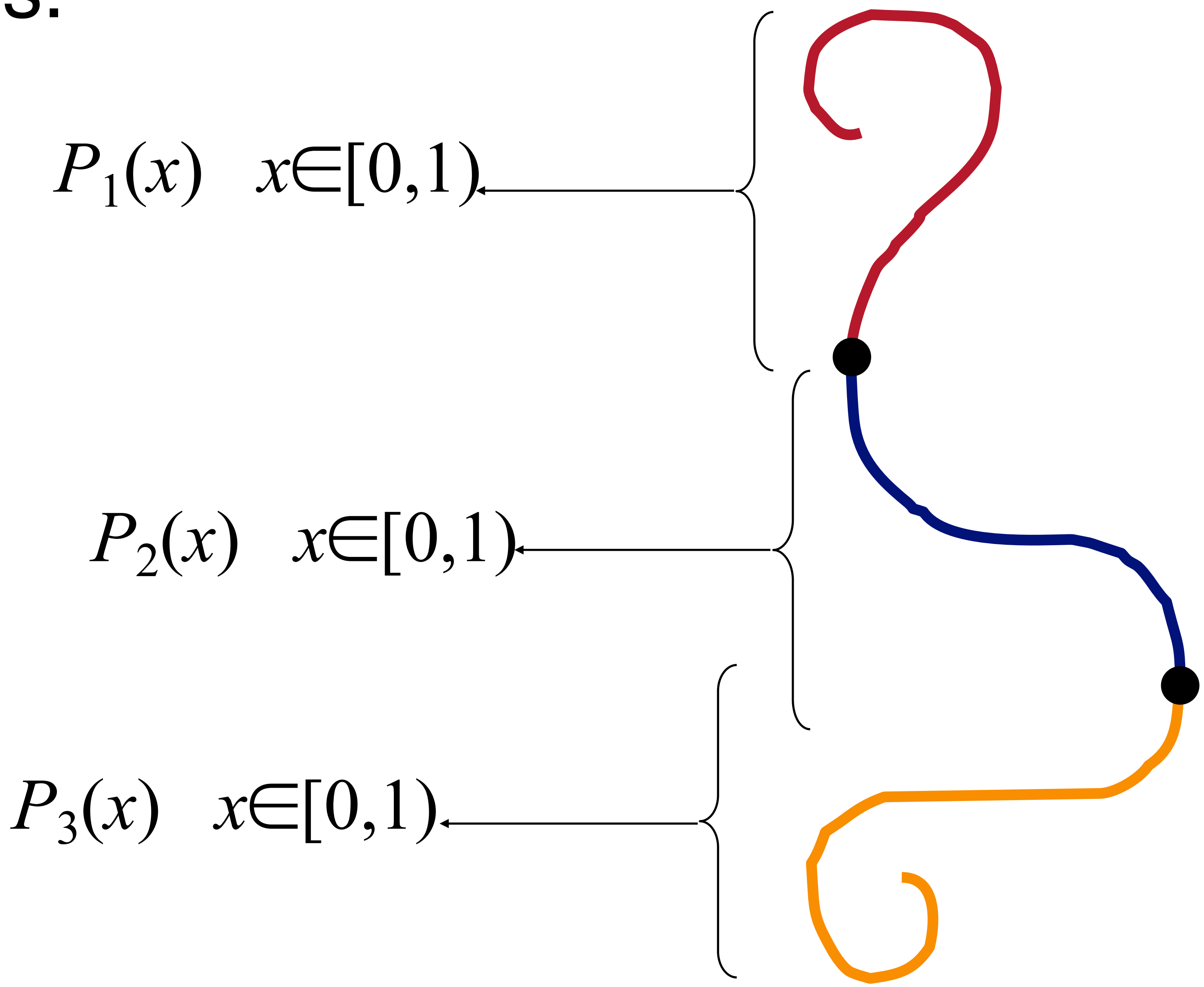
Geometric continuity: derivatives of the two curves are proportional (i.e. point in the same direction) where they meet.

- G^0 means two curves just meet
- G^1 means G^0 and 1st derivatives proportional
- G^2 means G^1 and 2nd derivatives proportional
- Parametric continuity used more frequently than geometric.



What is a Spline in CG?

A spline is a *piecewise* *polynomial function* whose derivatives satisfy some *continuity constraints* across curve boundaries.



$P_1(x) \quad x \in [0,1)$

$P_2(x) \quad x \in [0,1)$

$P_3(x) \quad x \in [0,1)$

$$\boxed{P_i(x) = \sum_{j=0}^{n} a_{ij} x^j}$$

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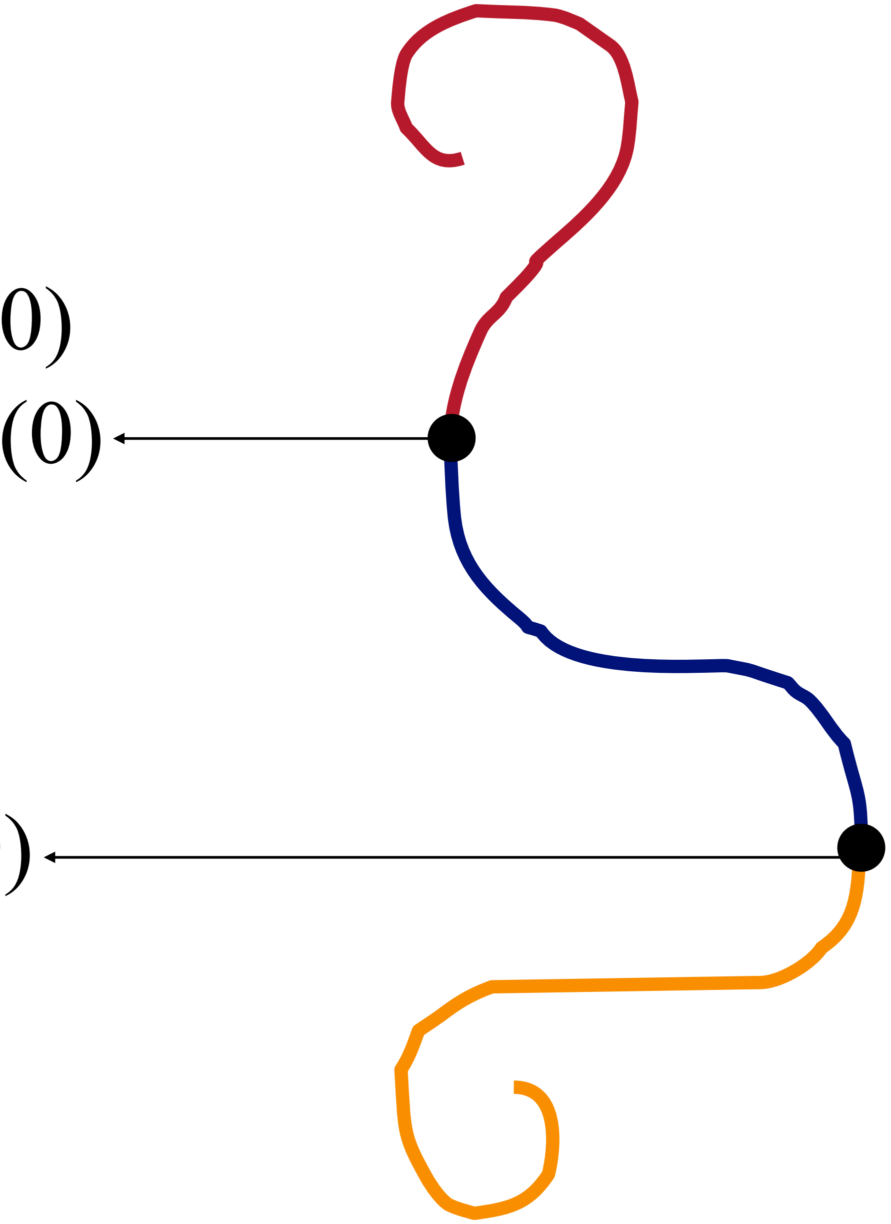

$P_1(1)=P_2(0)$

$P_1'(1)=P_2'(0)$

…

$P_2(1)=P_3(0)$

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…

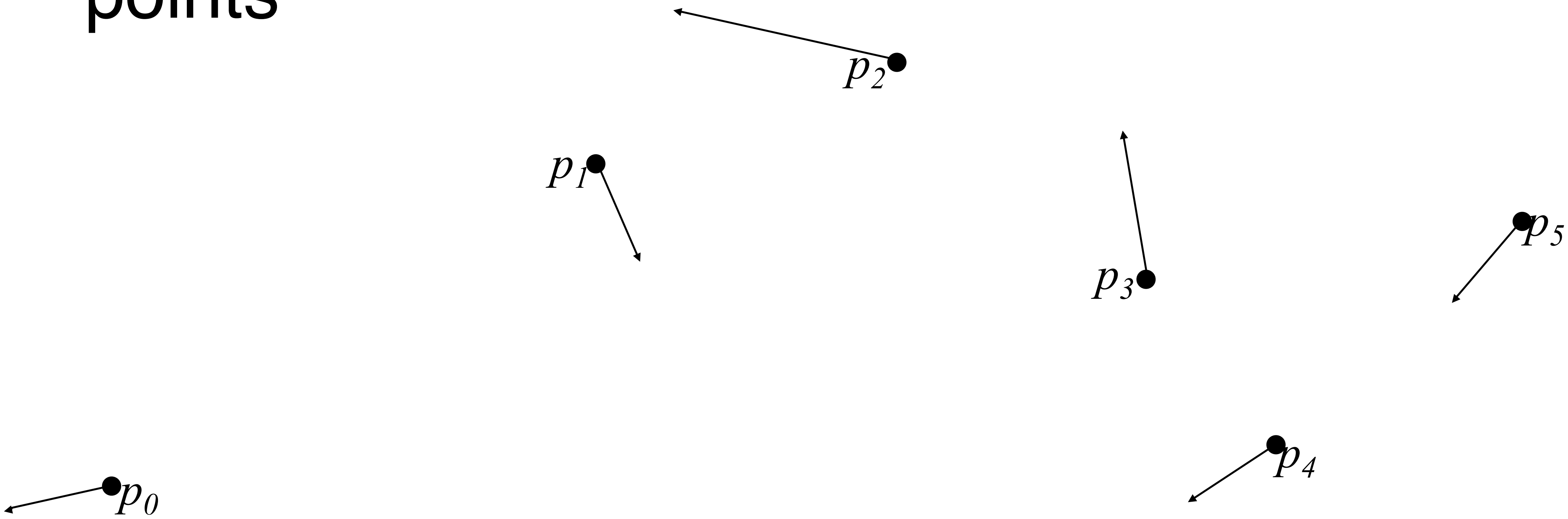
$$\boxed{P_i(x) = \sum_{j=0}^{n} a_{ij} x^j}$$

Overview

- What is a Spline?
- Specific Examples:
  - Hermite Splines
  - Cardinal Splines
  - Uniform Cubic B-Splines
- Comparing Cardinal Splines to Uniform Cubic B-Splines

Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial
- Specified with:
  - A pair of control points
  - Tangent at each control point
- Iteratively construct the curve between adjacent end points



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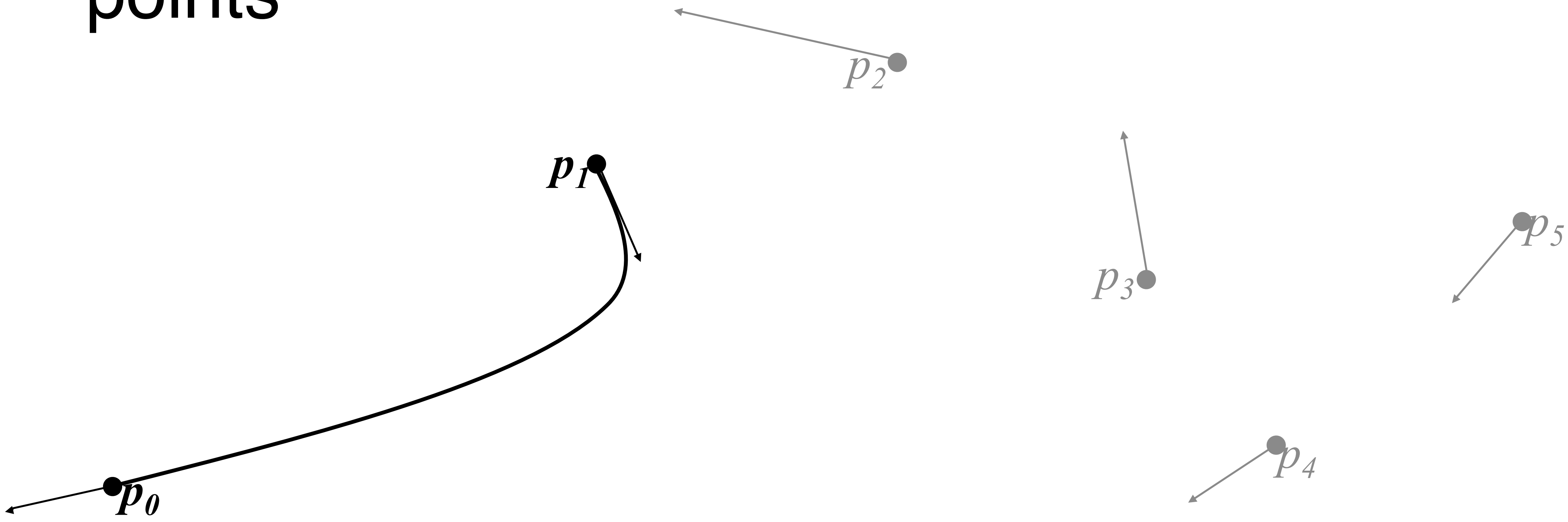

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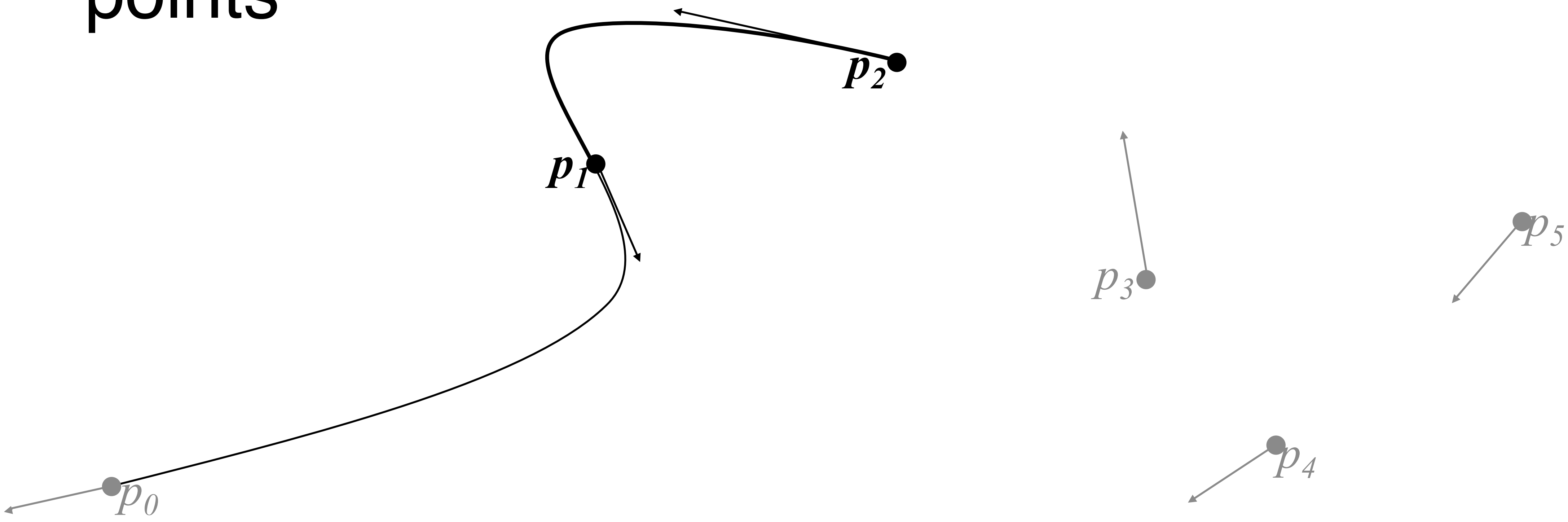

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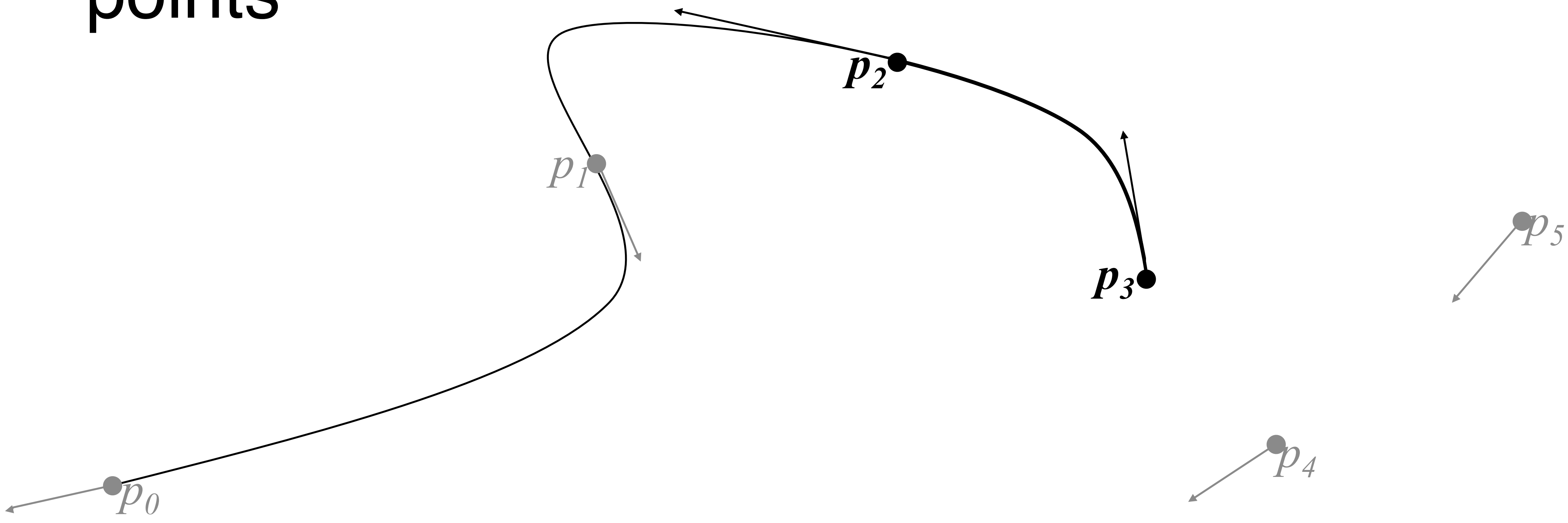

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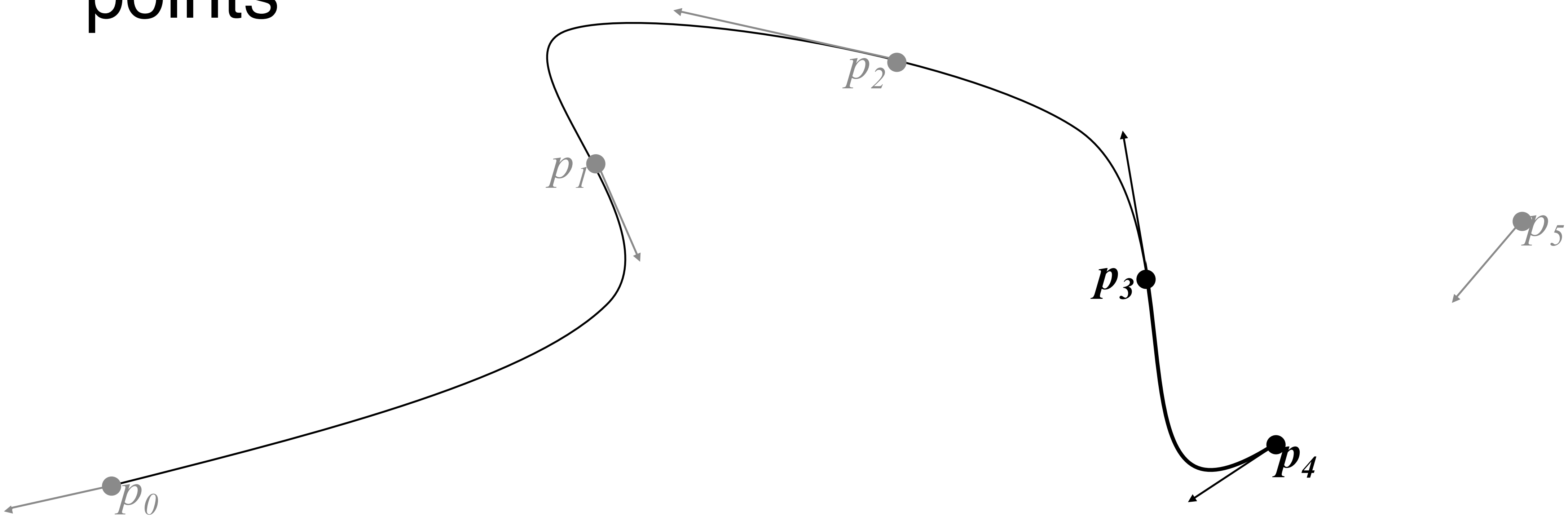

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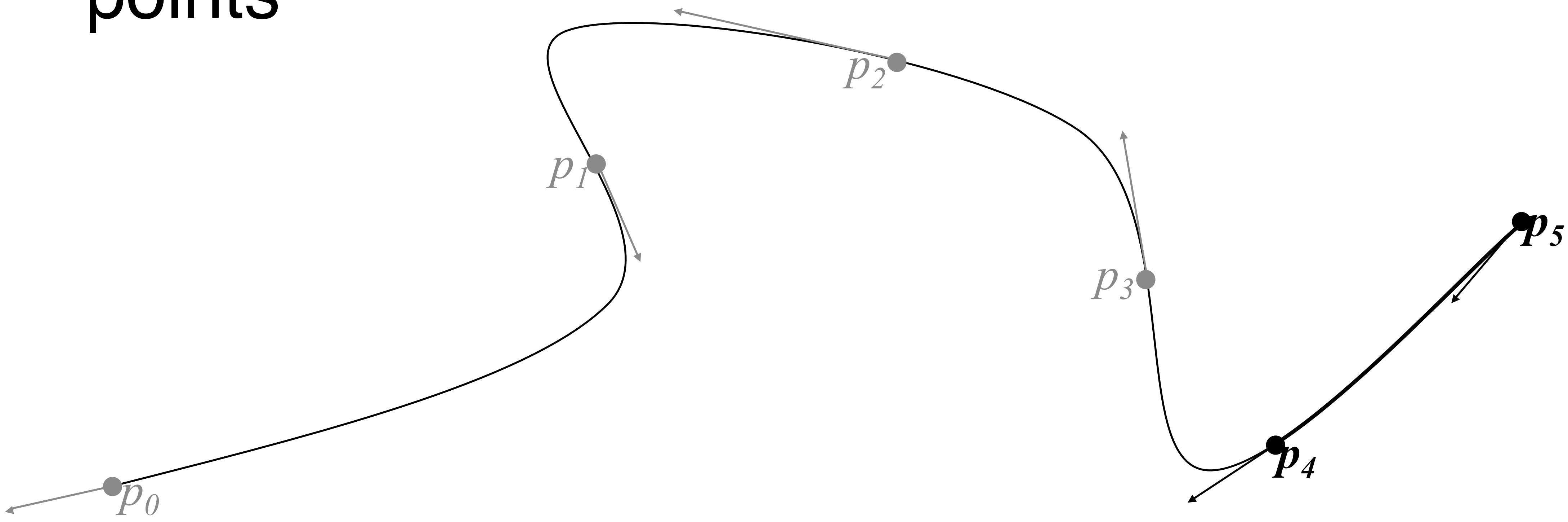

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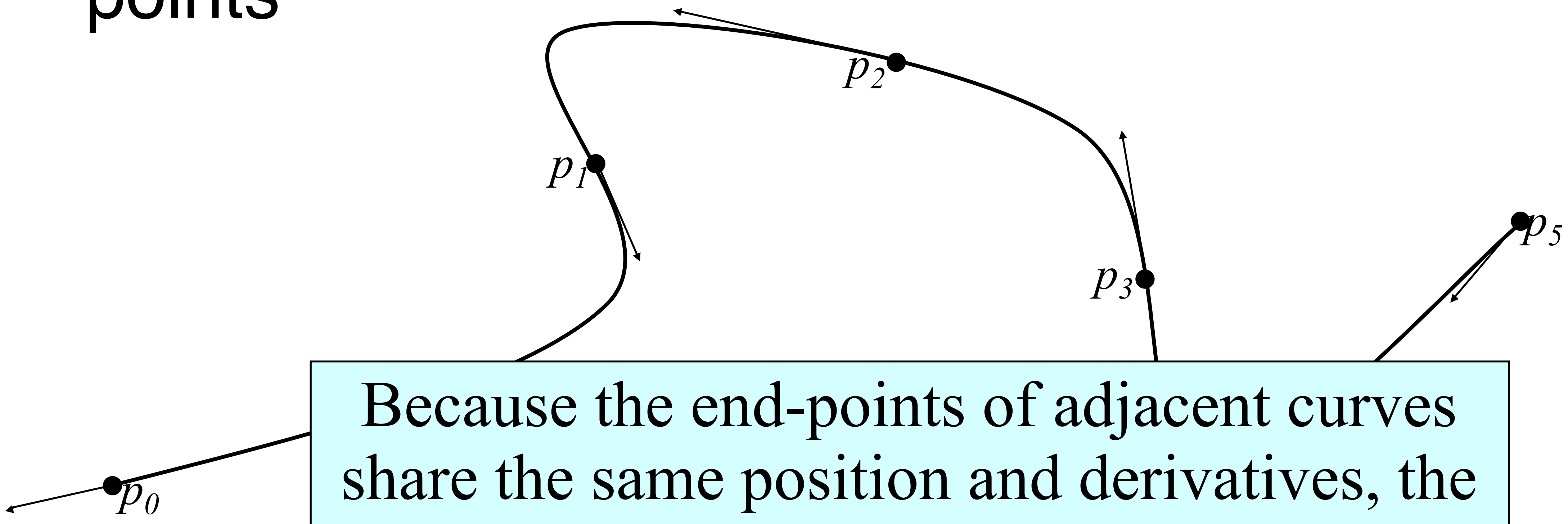
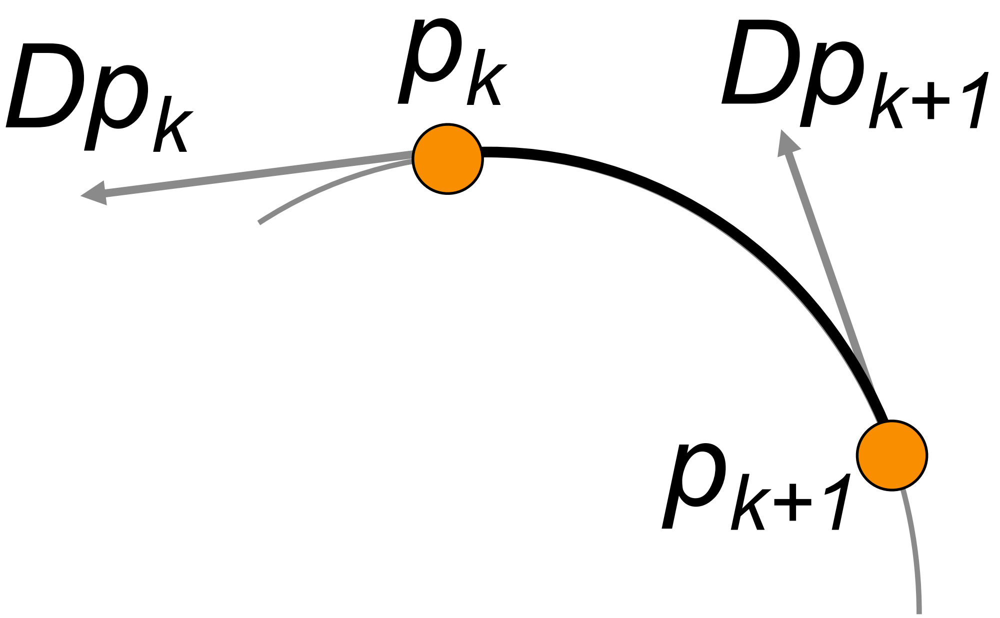


Because the end-points of adjacent curves share the same position and derivatives, the Hermite spline has C^1 continuity.

Specific Example: Hermite Splines

- Let $P_k(u)=(P_{k,X}(u),P_{k,Y}(u))$ with $0 \le u \le 1$ be a parametric cubic point function for the curve section between control points p_k and p_{k+1}

- Boundary conditions are:
  - $P_k(0) = p_k$
  - $P_k(1) = p_{k+1}$
  - $P_k'(0) = Dp_k$
  - $P_k'(1) = Dp_{k+1}$



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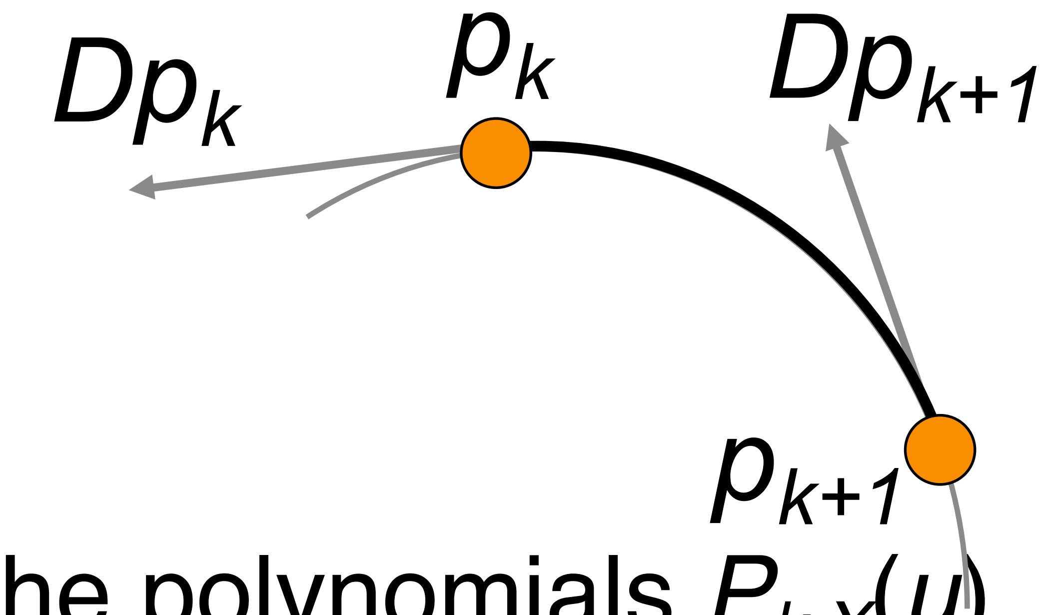

- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition

Specific Example: Hermite Splines

We can express the polynomials:

- $P(u) = au^3 + bu^2 + cu + d$
- $P'(u) = 3au^2 + 2bu + c$

using the matrix representations:

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad P'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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By abuse of notation, we will think of the coefficients a, b, c, and d as 2-vectors rather than scalars so that P is a function taking values in 2D.

Specific Example: Hermite Splines

Given the matrix representations:

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad P'(u) = \begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

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we can express the values at the end-points as:

$$p_k = P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad Dp_k = P'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$p_{k+1} = P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad Dp_{k+1} = P'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Specific Example: Hermite Splines

We can combine the equations

$$p_k = P(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad Dp_k = P'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$p_{k+1} = P(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad Dp_{k+1} = P'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

into a single matrix expression:

Specific Example: Hermite Splines

We can combine the equations

$$\mathrm{p}_{\mathrm{k}} = \mathrm{P}(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix} \qquad \mathrm{Dp}_{\mathrm{k}} = \mathrm{P}'(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix}$$

$$\mathrm{p}_{\mathrm{k}+1} = \mathrm{P}(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix} \qquad \mathrm{Dp}_{\mathrm{k}+1} = \mathrm{P}'(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix}$$

into a single matrix expression:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Specific Example: Hermite Splines

Inverting the matrix in the equation:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

we get:

Specific Example: Hermite Splines

Inverting the matrix in the equation:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

we get:

$$\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{p_k} \\ \mathrm{p_{k+1}} \\ \mathrm{Dp_k} \\ \mathrm{Dp_{k+1}} \end{bmatrix}$$

Specific Example: Hermite Splines

Inverting the matrix in the equation:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

we get:

$$\begin{bmatrix} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{p}_\mathrm{k} \\ \mathrm{p}_{\mathrm{k}+1} \\ \mathrm{Dp}_\mathrm{k} \\ \mathrm{Dp}_{\mathrm{k}+1} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{p}_\mathrm{k} \\ \mathrm{p}_{\mathrm{k}+1} \\ \mathrm{Dp}_\mathrm{k} \\ \mathrm{Dp}_{\mathrm{k}+1} \end{bmatrix}$$

Specific Example: Hermite Splines

Using the facts that:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} \quad \text{and} \quad P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

we get:

Specific Example: Hermite Splines

Using the facts that:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix} \quad \text{and} \quad P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

we get:

$$P(u) = \underbrace{\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}}_{\text{parameters}} \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{M_{\text{Hermite}}} \underbrace{\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}}_{\text{boundary info}}$$

Specific Example: Hermite Splines

and we can execute matrix multiplies below

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}$$

to get

$$P(u) = p_k(2u^3 - 3u^2 + 1) + p_{k+1}(-2u^3 + 3u^2) + \\ Dp_k(u^3 - 2u^2 + u) + Dp_{k+1}(u^3 - u^2)$$

Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

we can re-write the equation:

$$P(u) = p_k(2u^3 - 3u^2 + 1) + p_{k+1}(-2u^3 + 3u^2) + Dp_k(u^3 - 2u^2 + u) + Dp_{k+1}(u^3 - u^2)$$

as:

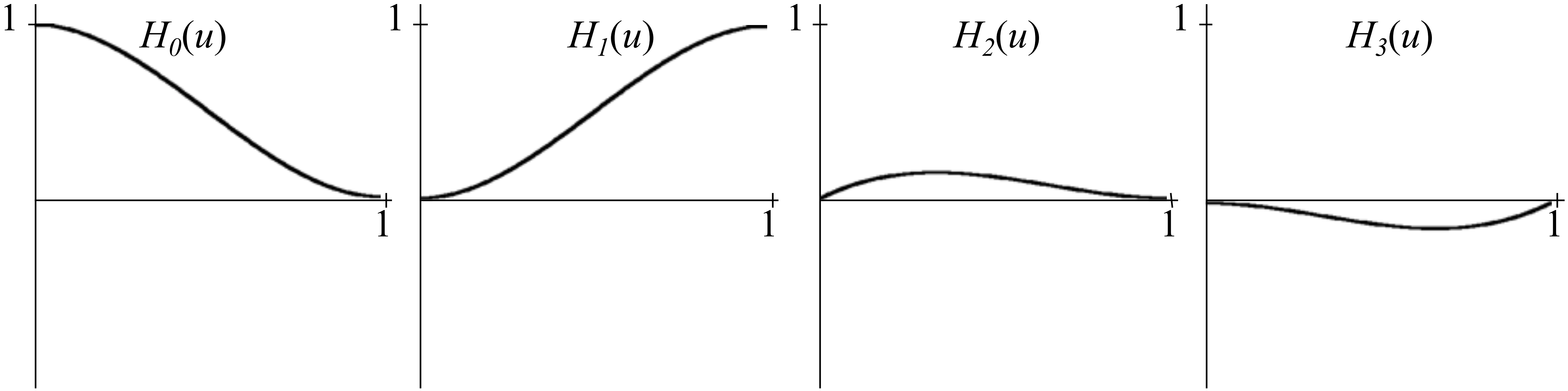
$$P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)$$

Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
- $H_3(u) = u^3 - u^2$

Blending Functions



$$P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)$$

Specific Example: Hermite Splines

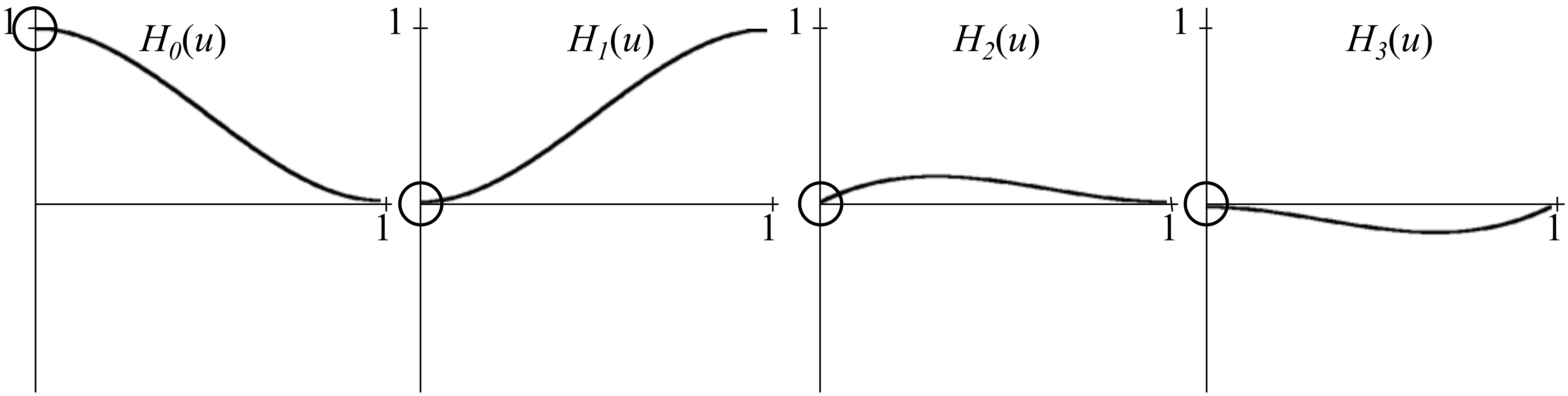
Setting:

- $H_0(u)= 2u^3-3u^2+1$
- $H_1(u)=-2u^3+3u^2$
- $H_2(u)= u^3-2u^2+u$
- $H_3(u)= u^3-u^2$

When $u=0$:

- $H_0(u)=1$
- $H_1(u)=0$
- $H_2(u)=0$
- $H_3(u)=0$

So $P(0)=p_k$



$$P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)$$

Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
- $H_1(u) = -2u^3 + 3u^2$
- $H_2(u) = u^3 - 2u^2 + u$
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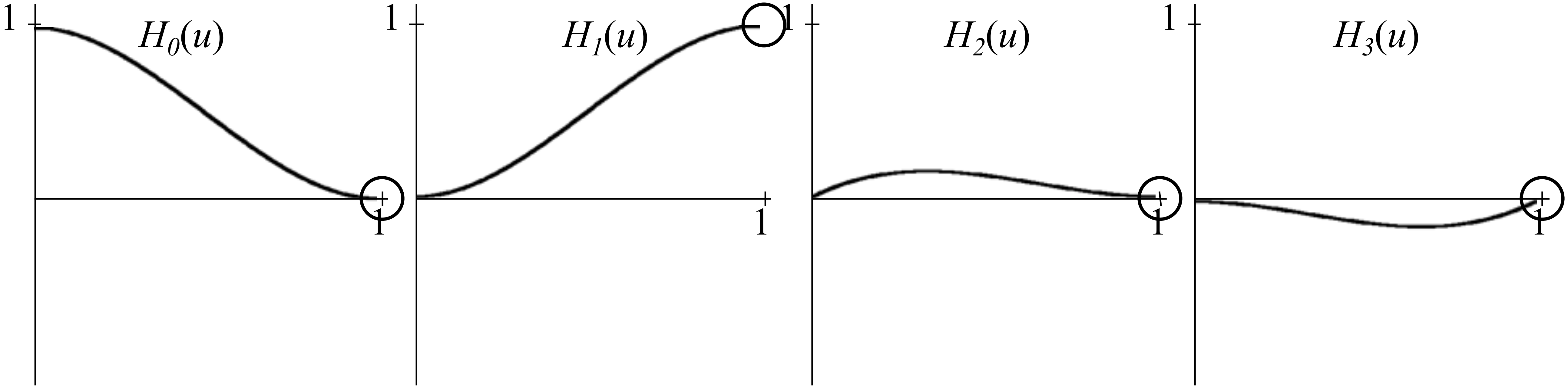
When $u=1$:

- $H_0(u)=0$
- $H_1(u)=1$
- $H_2(u)=0$
- $H_3(u)=0$

So $P(1)=p_{k+1}$



$$P(u) = p_k H_0(u) + p_{k+1} H_1(u) + Dp_k H_2(u) + Dp_{k+1} H_3(u)$$

Specific Example: Hermite Splines

Setting:

- $H_0(u) = 2u^3 - 3u^2 + 1$
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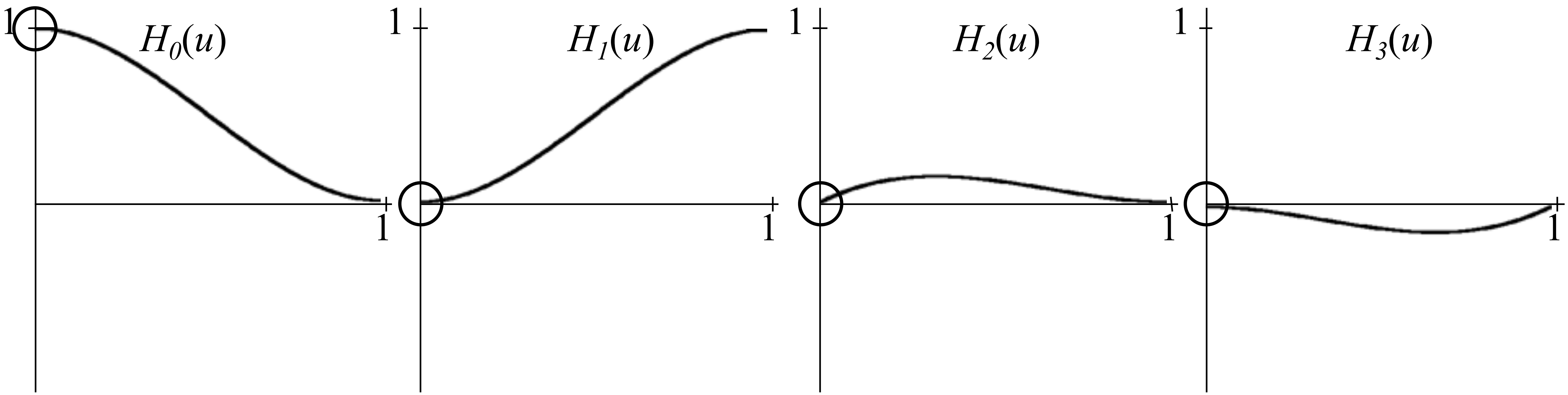
- $H_3(u) = u^3 - u^2$

When $u=0$:

- $H_0'(u)=0$
- $H_1'(u)=0$
- $H_2'(u)=1$
- $H_3'(u)=0$

So $P'(0)=Dp_k$



$$P'(u) = p_k H_0'(u) + p_{k+1} H_1'(u) + Dp_k H_2'(u) + Dp_{k+1} H_3'(u)$$

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Setting:

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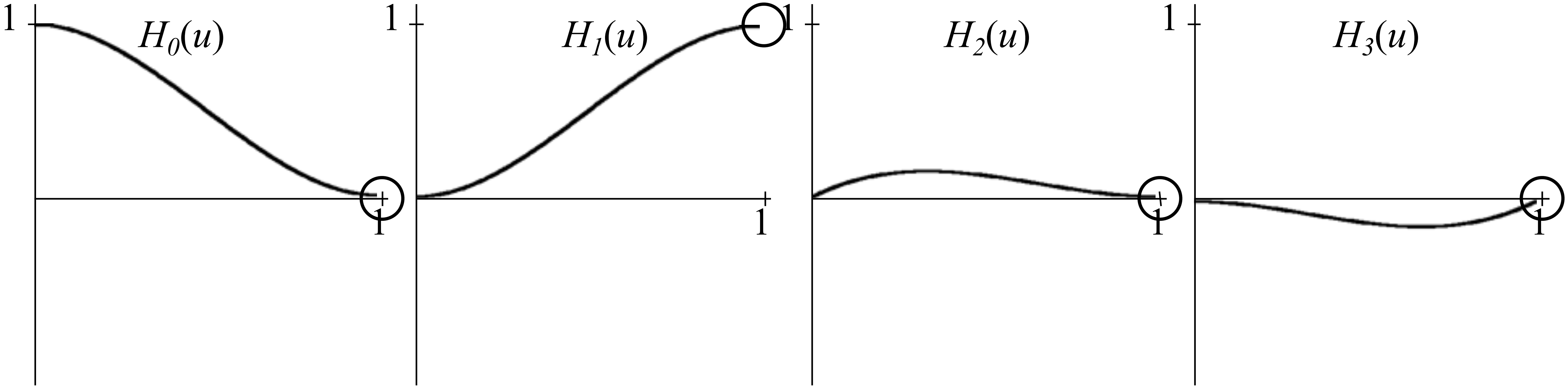
When $u=1$:

- $H_0'(u)=0$
- $H_1'(u)=0$
- $H_2'(u)=0$
- $H_3'(u)=1$

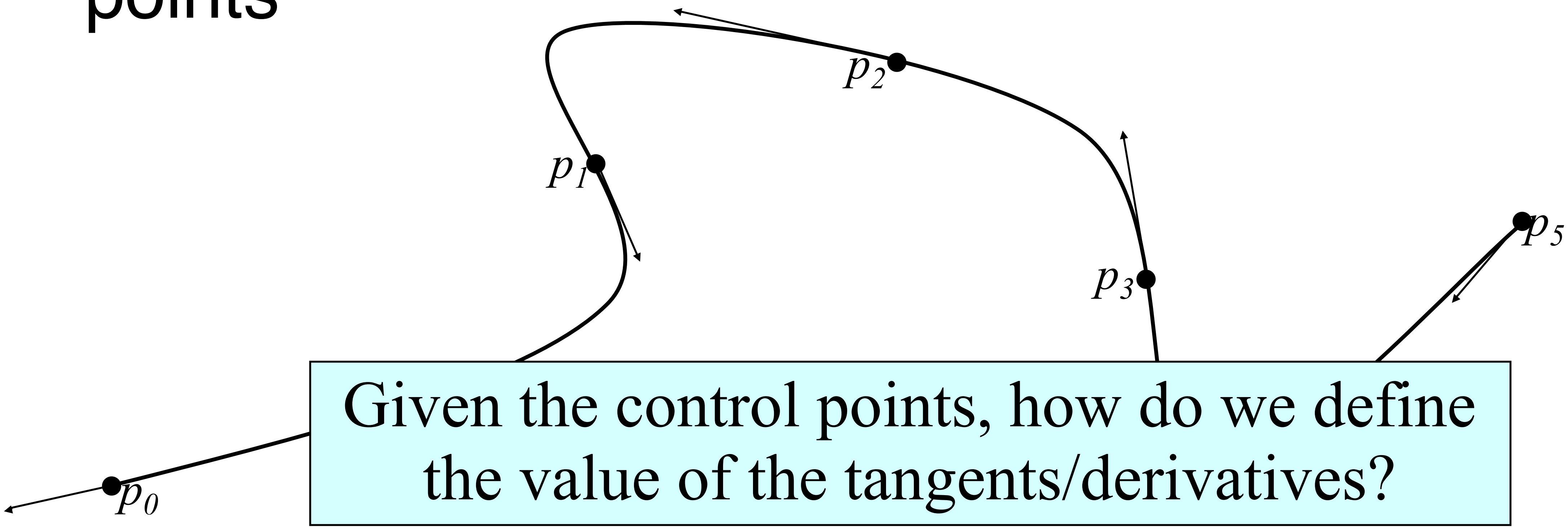
So $P'(1)=Dp_{k+1}$



$$P'(u) = p_k H_0'(u) + p_{k+1} H_1'(u) + Dp_k H_2'(u) + Dp_{k+1} H_3'(u)$$

Specific Example: Hermite Splines

- Interpolating piecewise *cubic* polynomial
- Specified with:
  - Set of control points
  - Tangent at each control point
- Iteratively construct the curve between adjacent end points



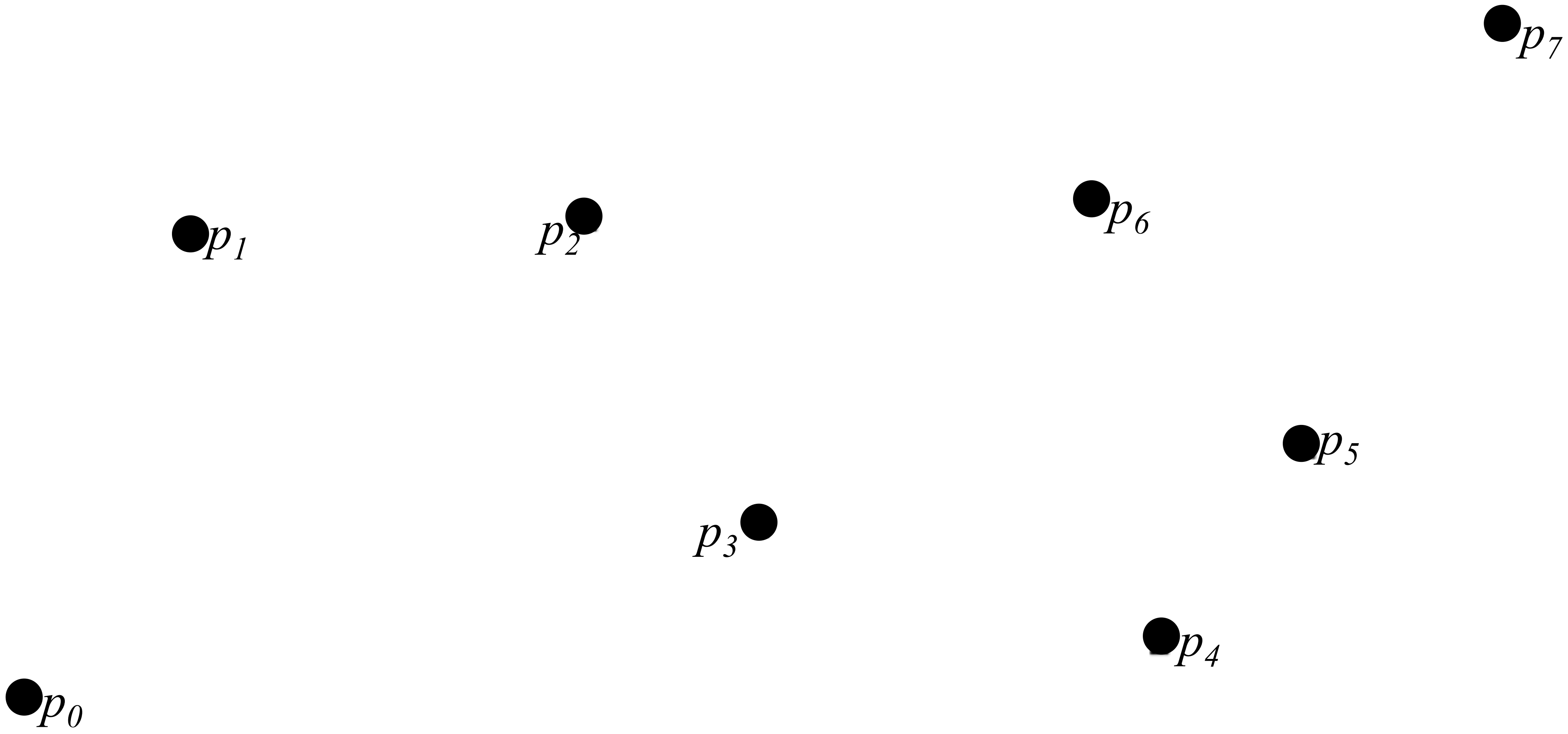
Given the control points, how do we define the value of the tangents/derivatives?

Overview

- What is a Spline?
- Specific Examples:
  - Hermite Splines
  - Cardinal Splines
  - Uniform Cubic B-Splines
- Comparing Cardinal Splines to Uniform Cubic B-Splines

Specific Example: Cardinal Splines

- Interpolating piecewise *cubic* polynomial
- Specified with four control points
- Iteratively construct the curve between middle two points using adjacent points to define tangents



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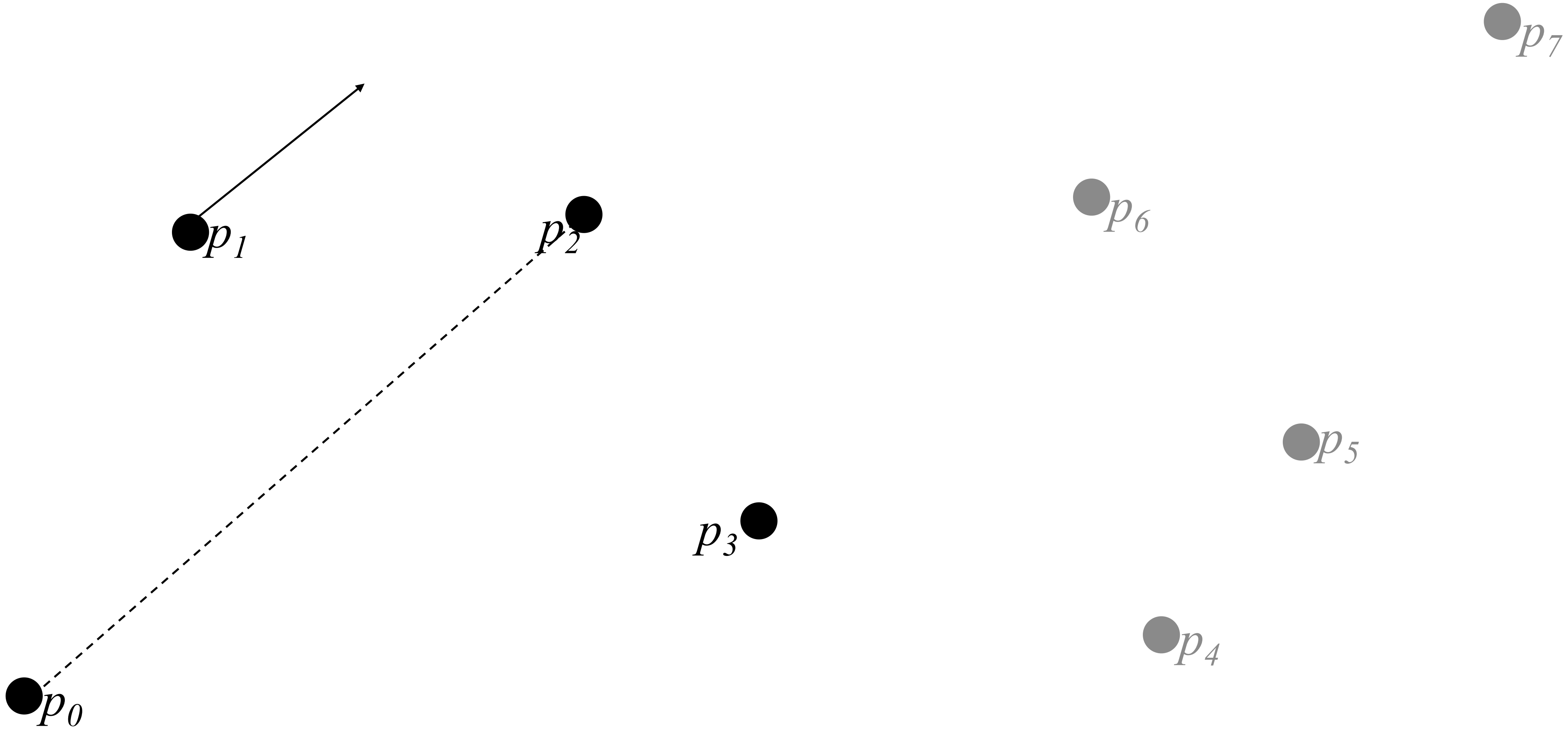

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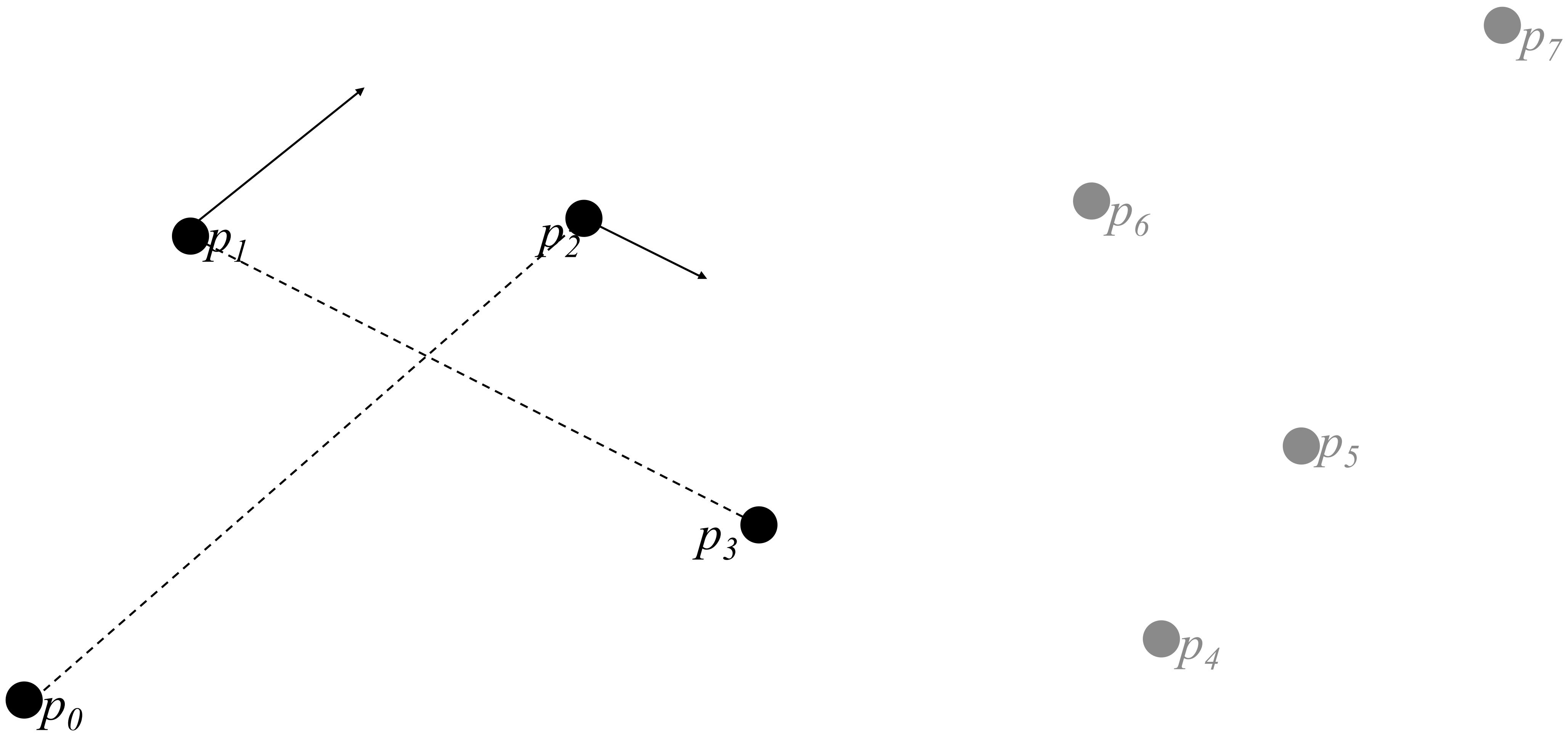

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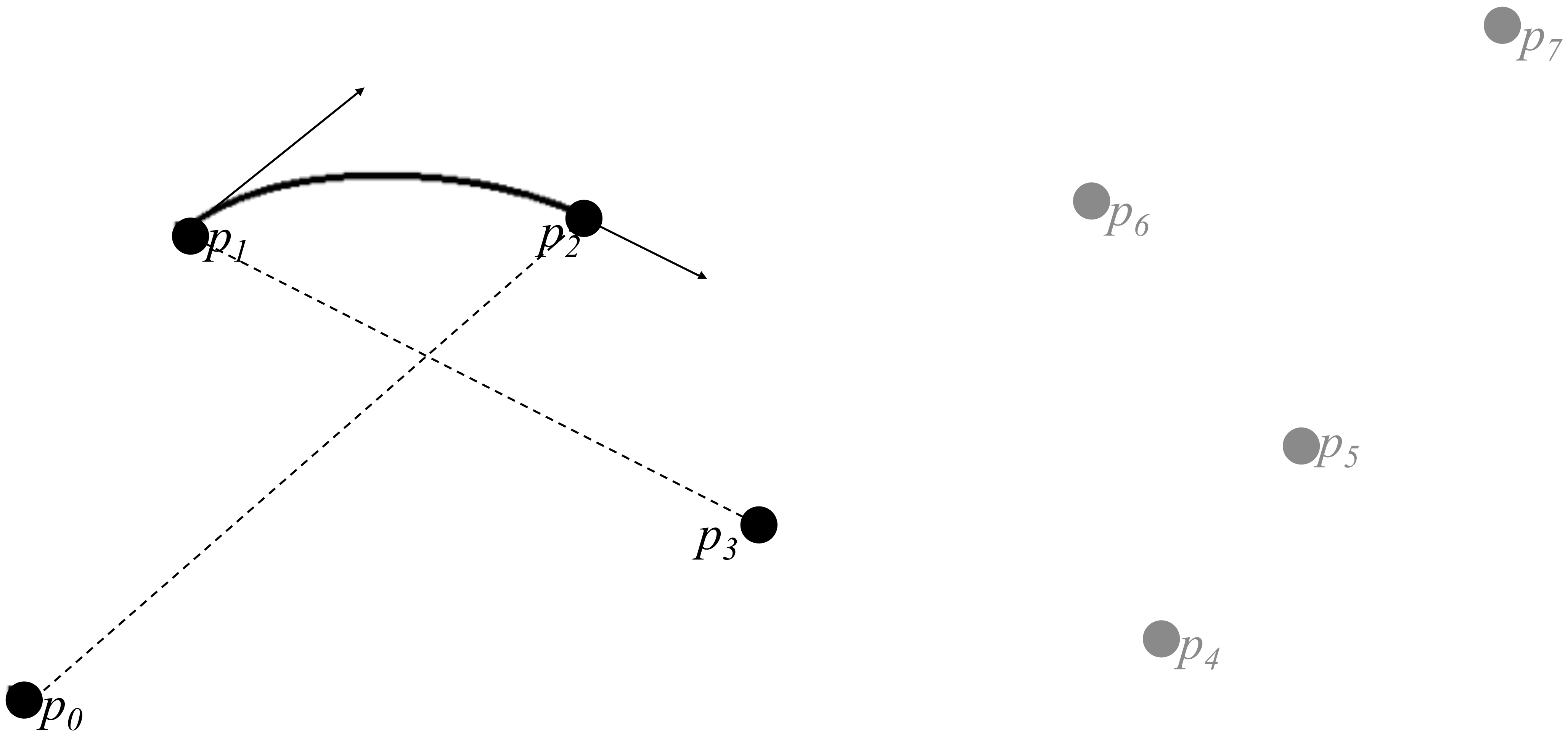

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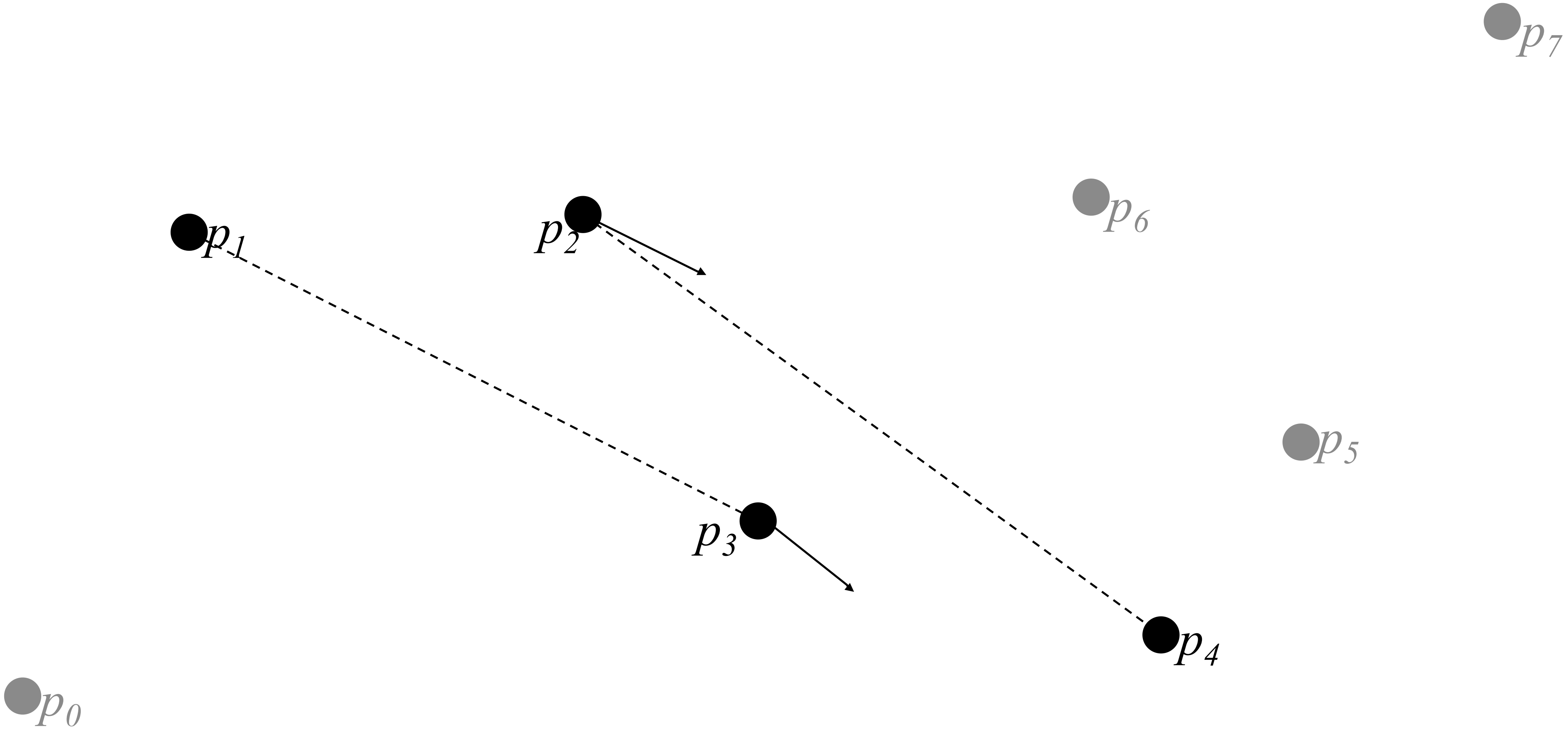

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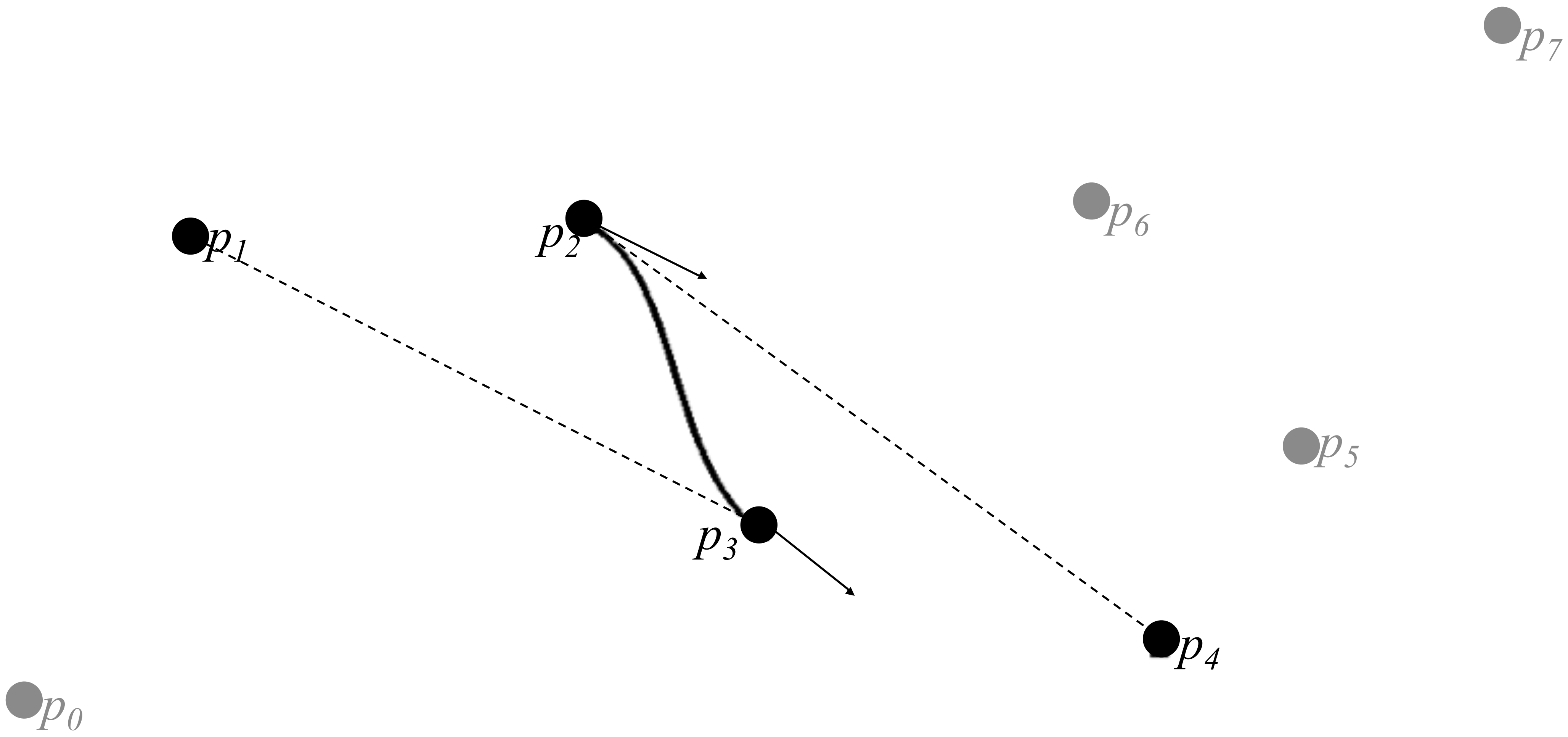

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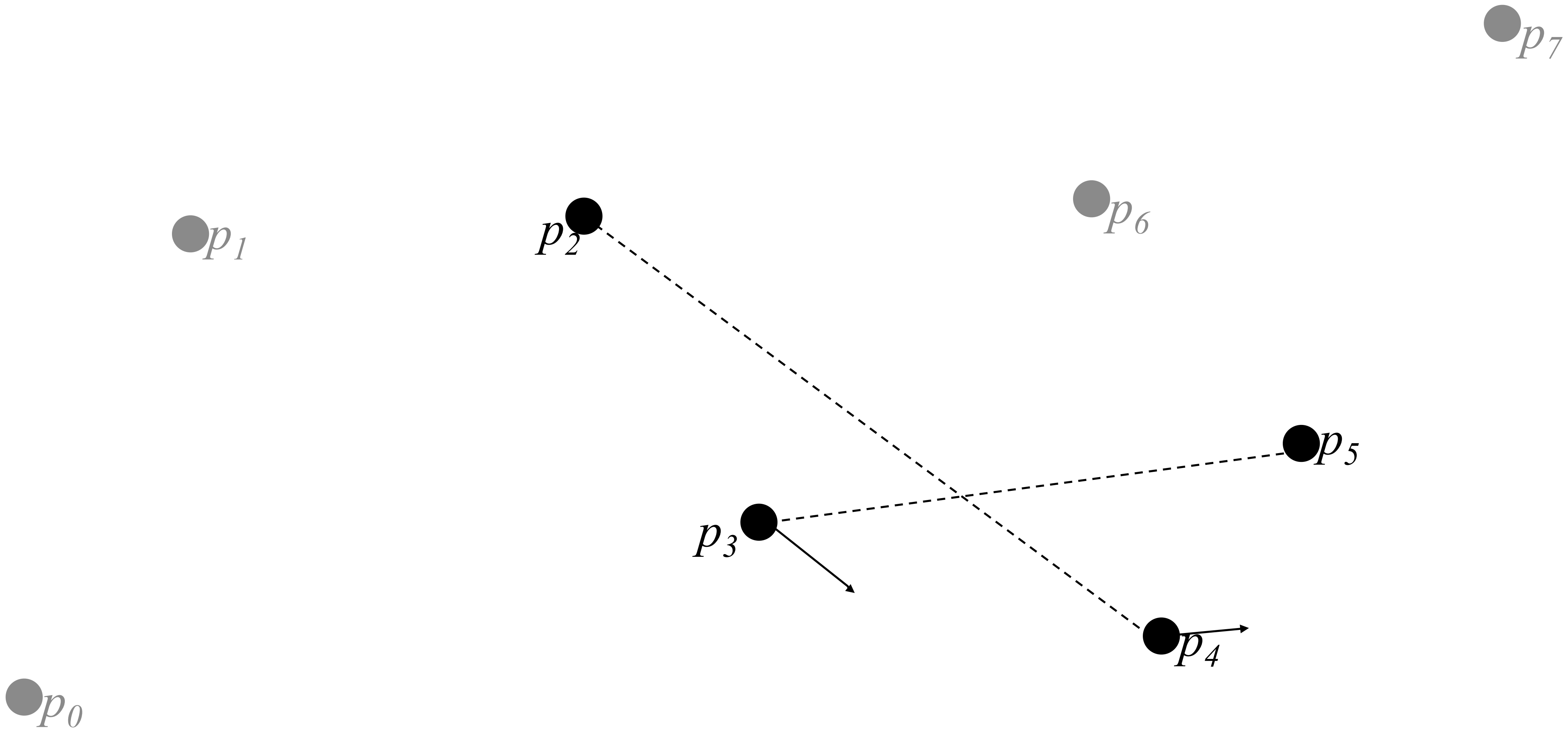

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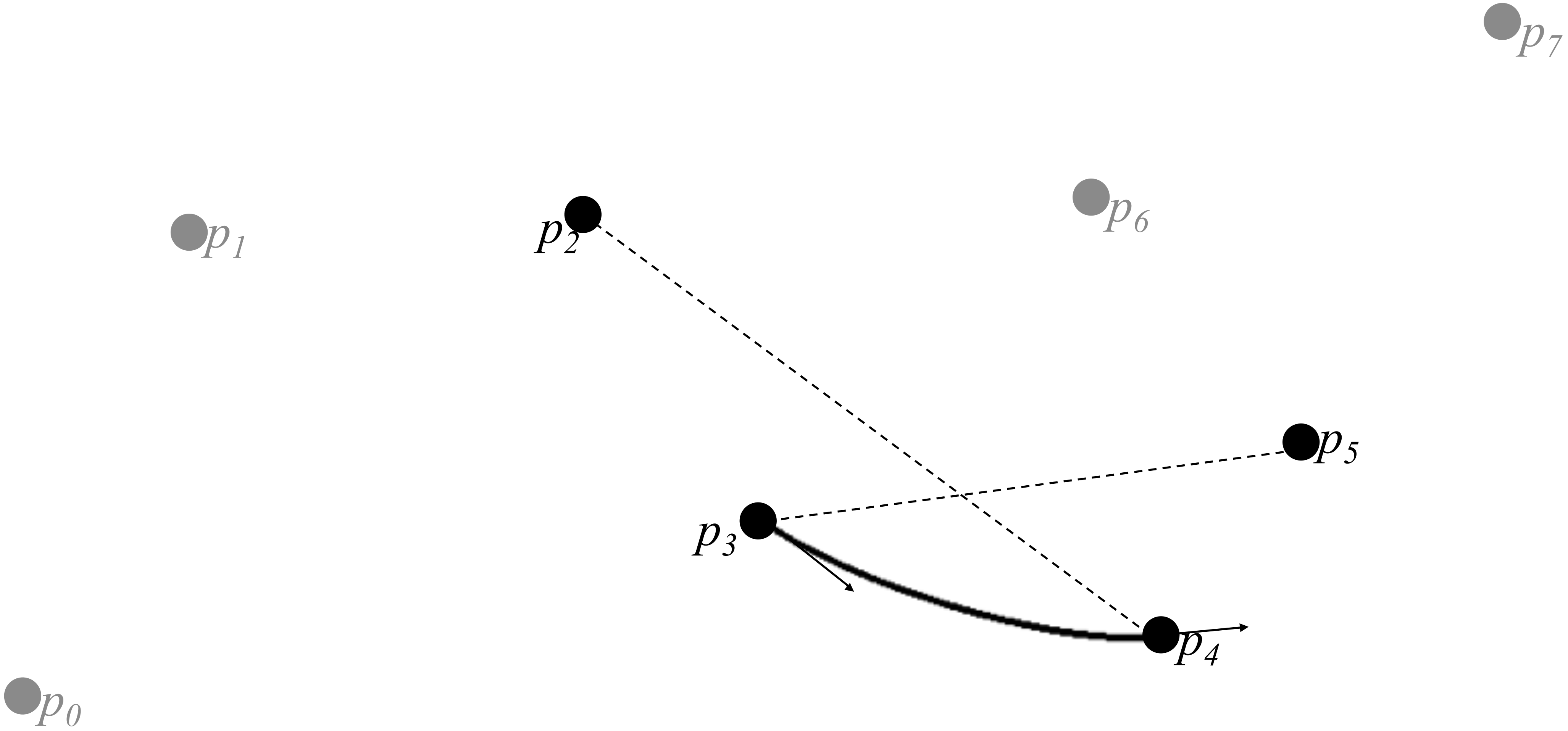

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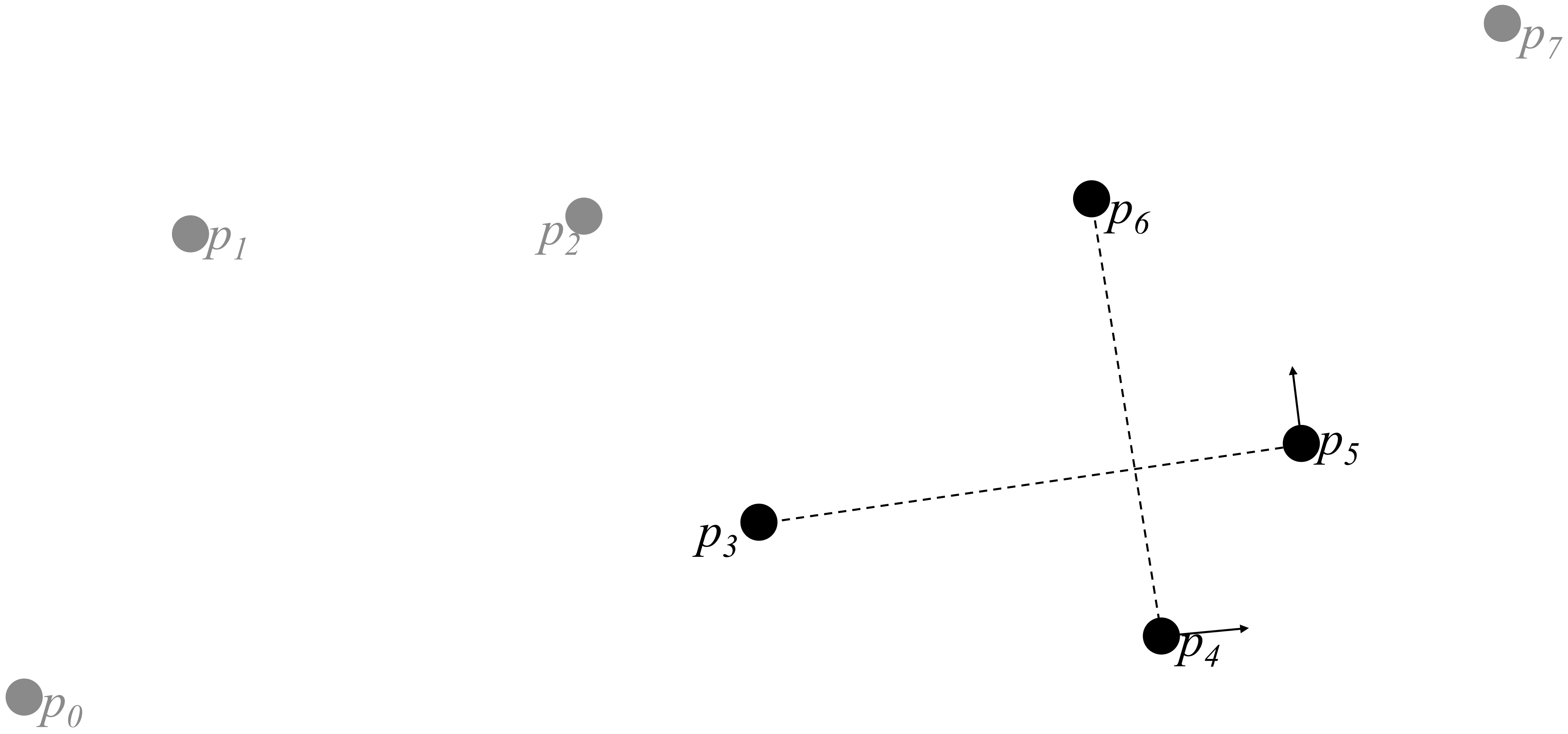

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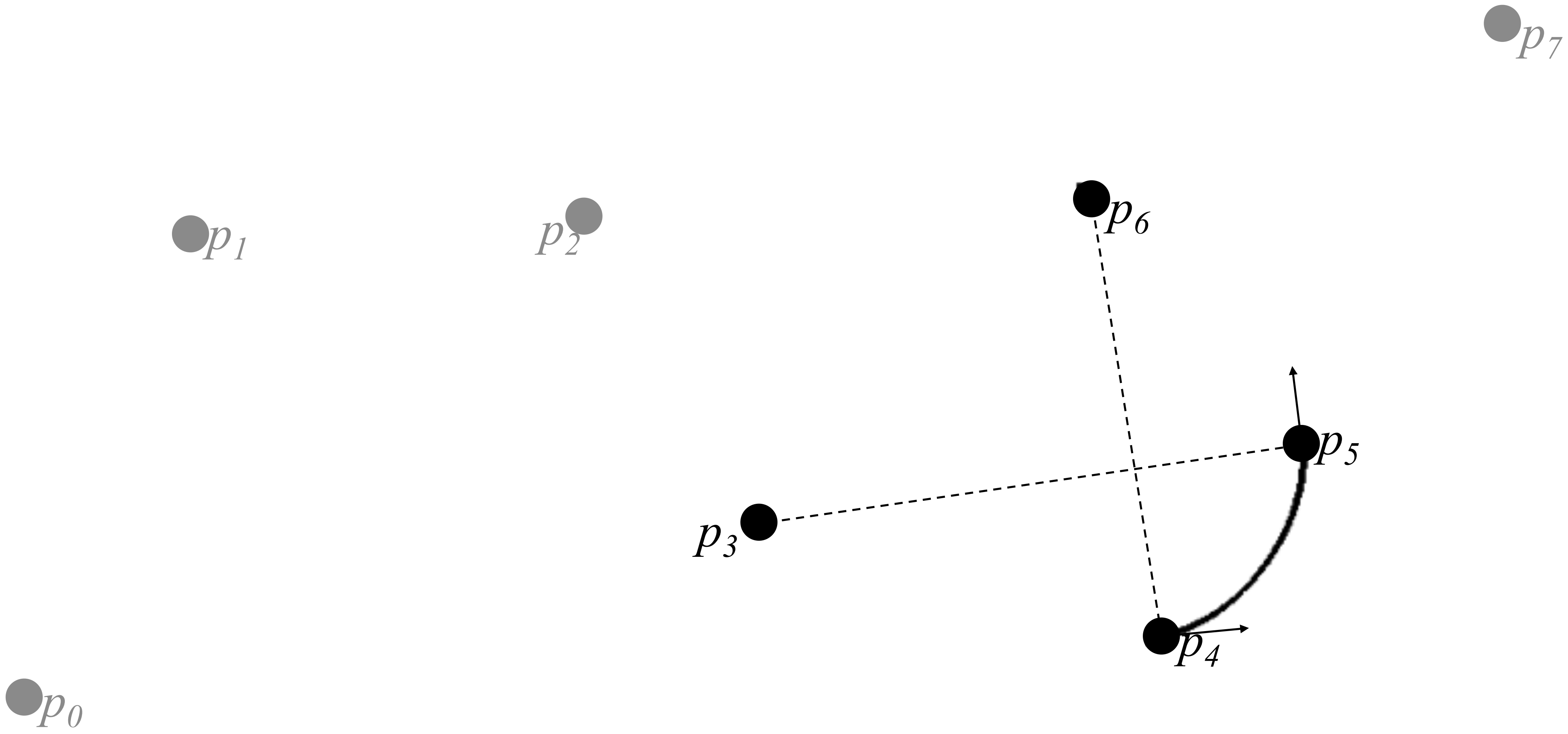

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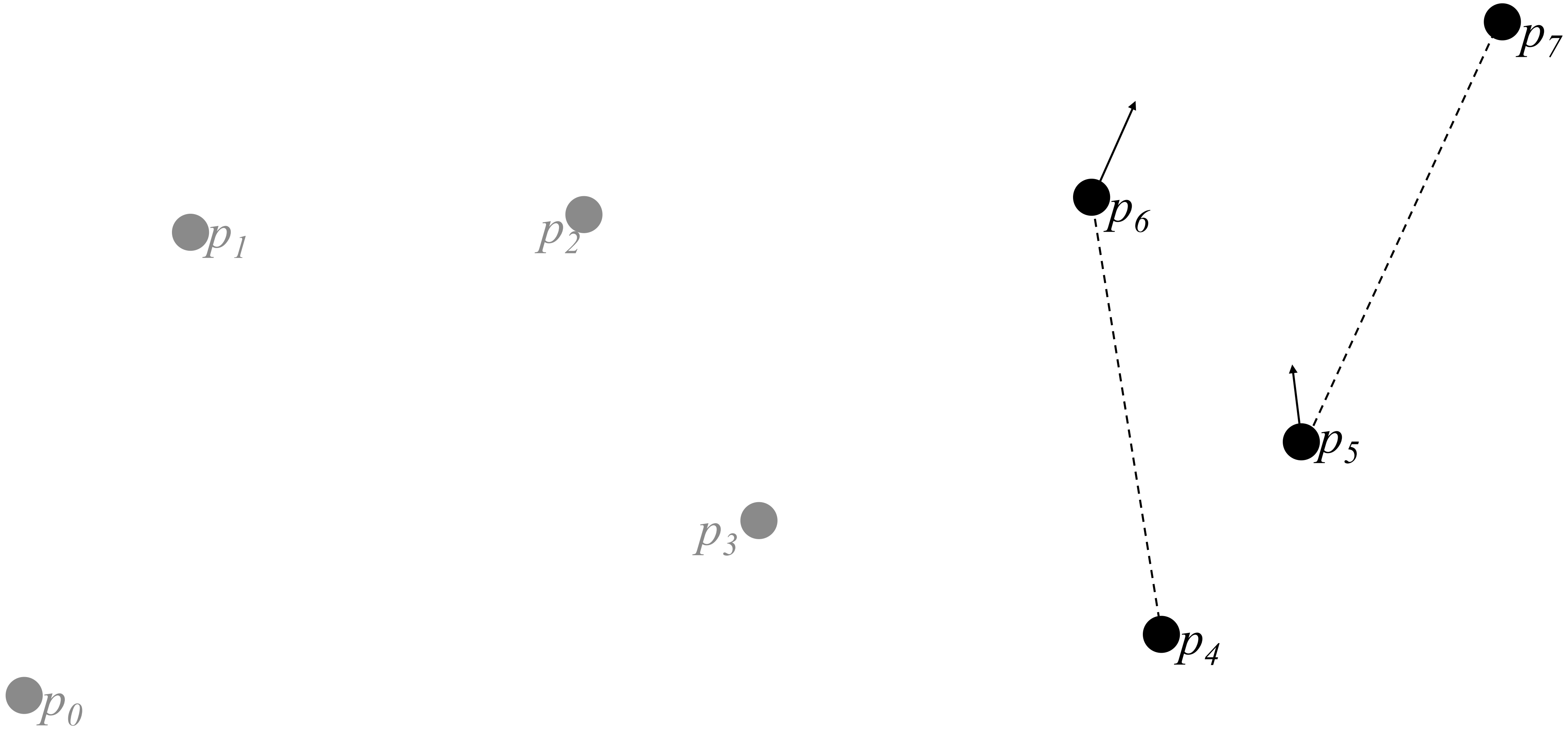

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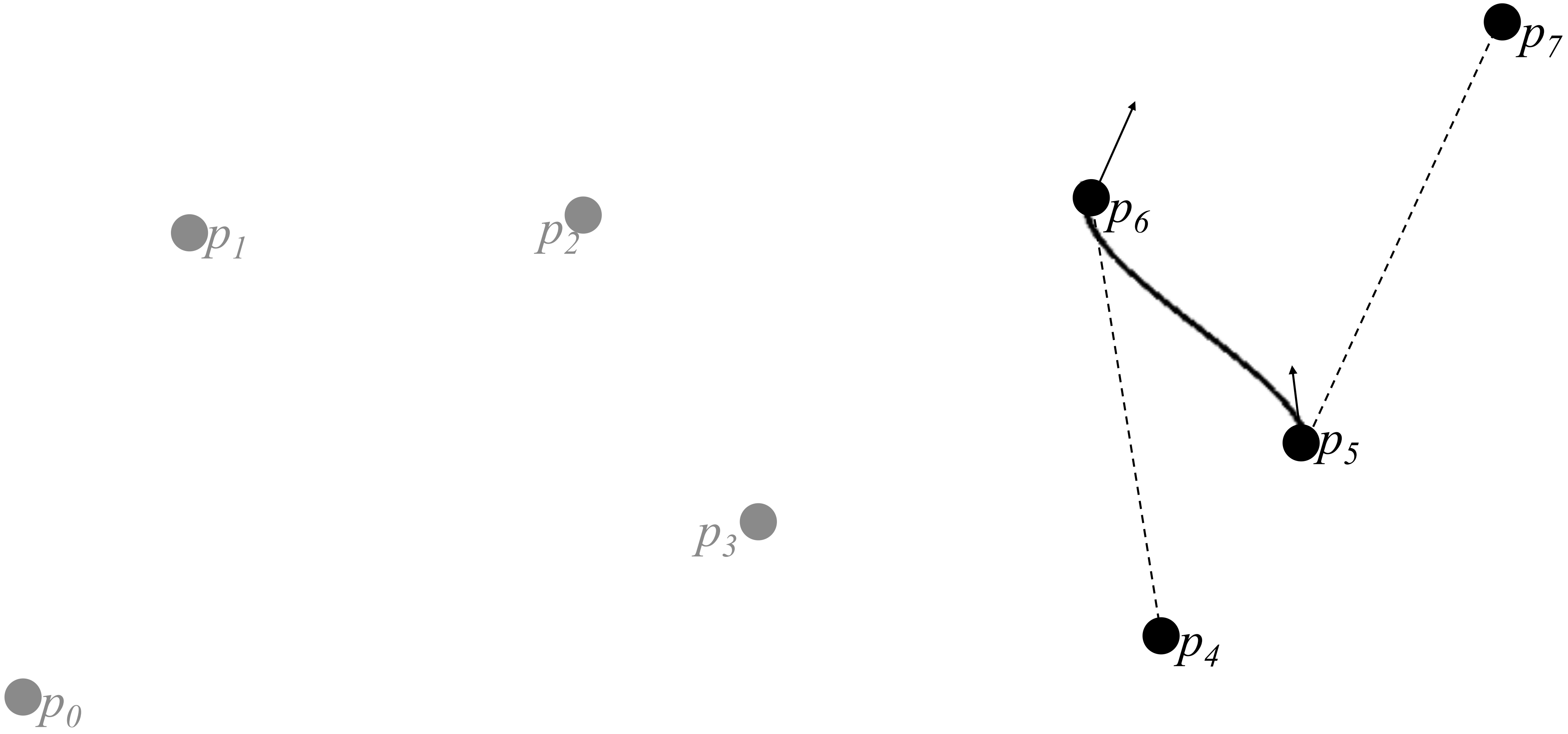

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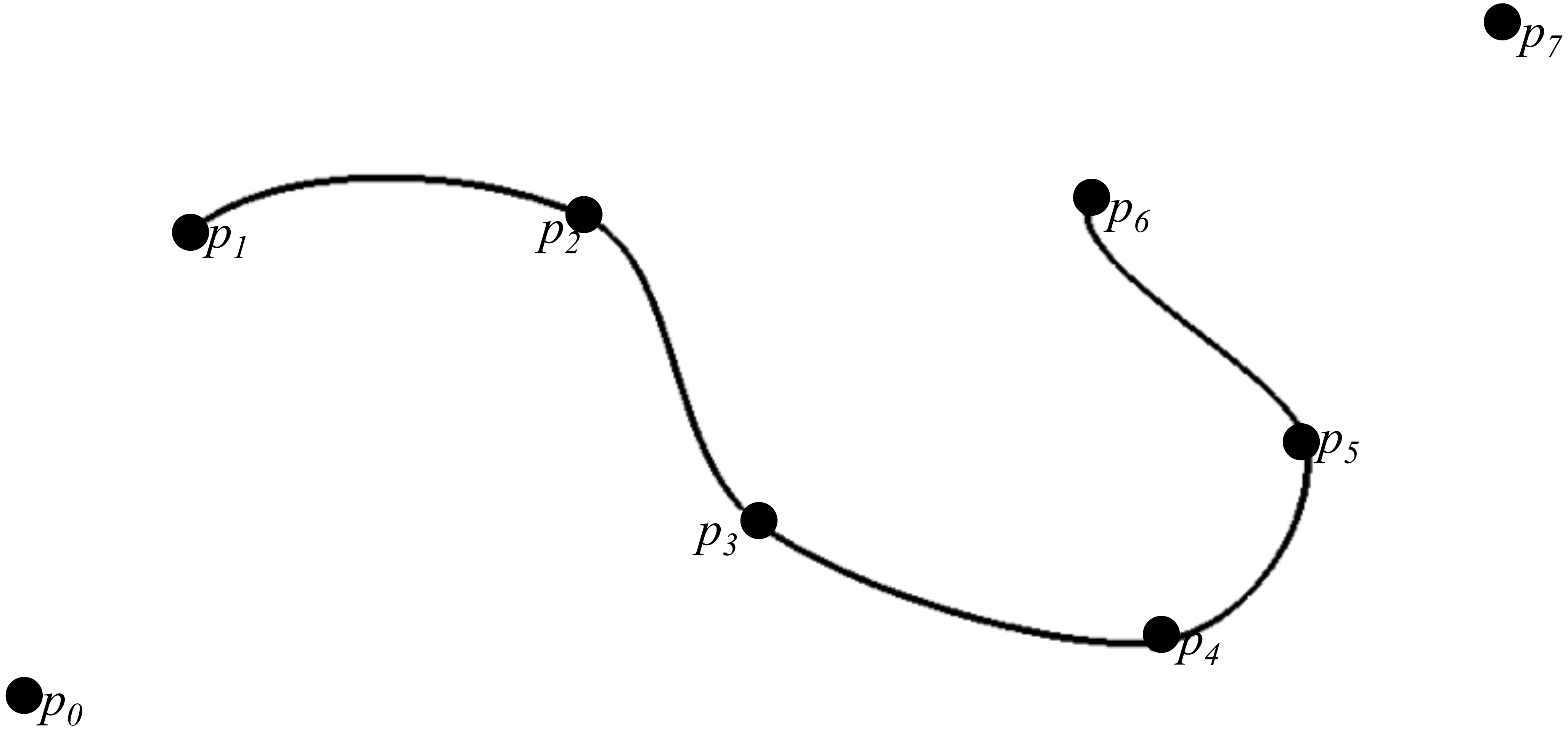

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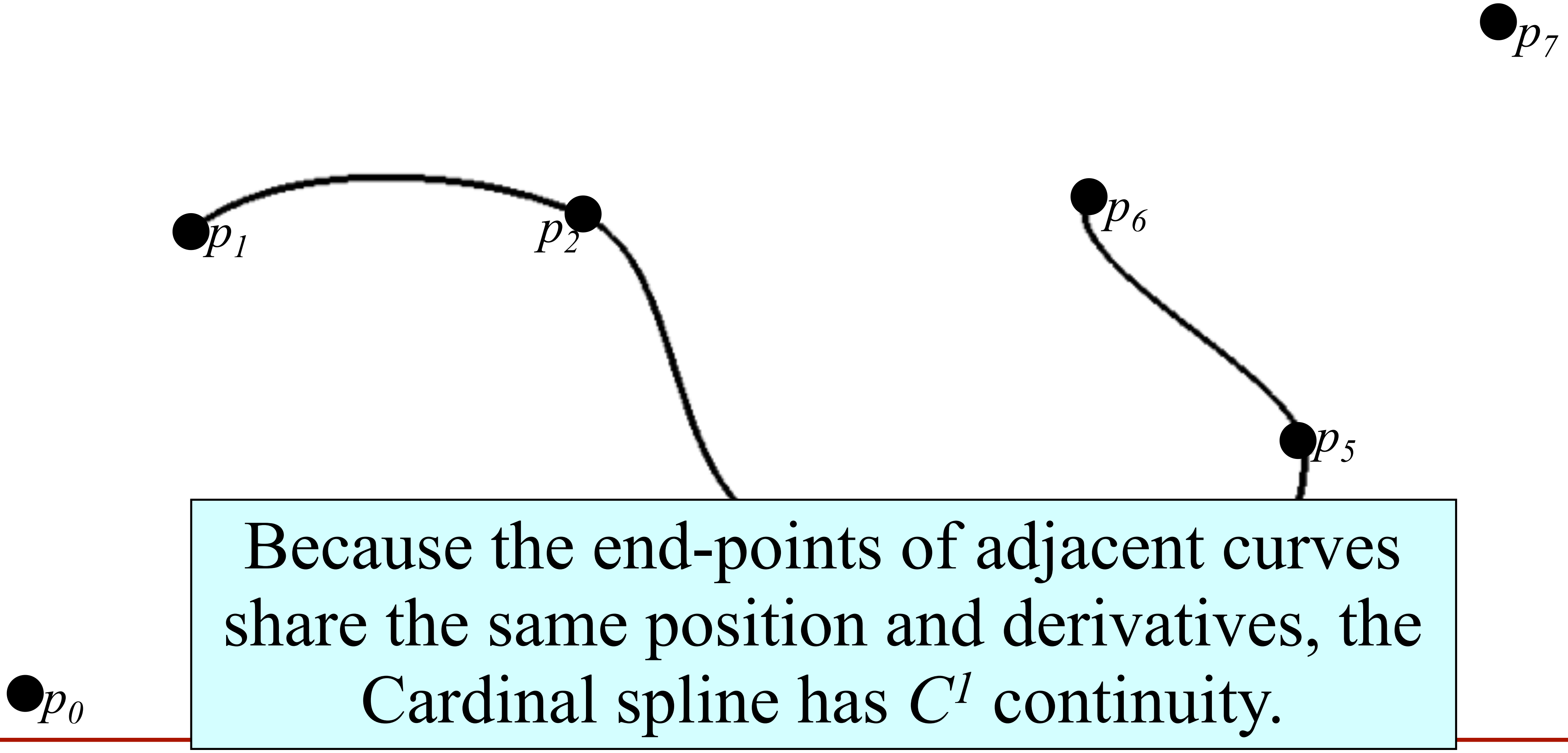
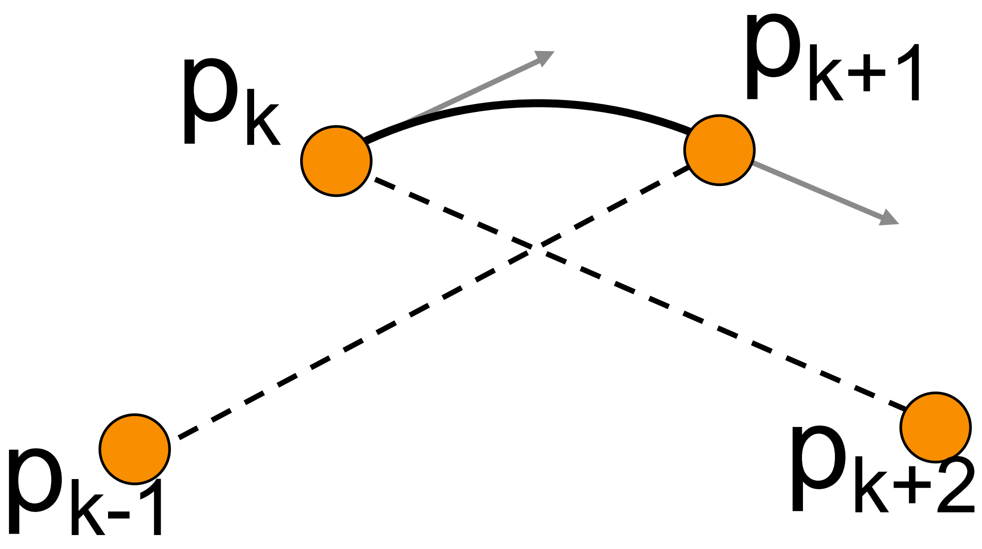


Because the end-points of adjacent curves share the same position and derivatives, the Cardinal spline has C^1 continuity.

Specific Example: Cardinal Splines

- Let $P_k(u)=(P_{k,X}(u),P_{k,Y}(u))$ with $0 \le u \le 1$ be a parametric cubic point function for the curve section between control points p_k and p_{k+1}

- Boundary conditions are:
  - $P(0) = p_k$
  - $P(1) = p_{k+1}$
  - $P'(0) = \frac{1}{2}(1 - t)(p_{k+1} - p_{k-1})$
  - $P'(1) = \frac{1}{2}(1 - t)(p_{k+2} - p_k)$



- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition

Specific Example: Cardinal Splines

Recall:

The Hermite matrix determines the coefficients of the polynomial from the positions and the derivatives of the end-points

$$P(u) = \underbrace{\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}}_{\text{parameters}} \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{M_{\text{Hermite}}} \underbrace{\begin{bmatrix} p_k \\ p_{k+1} \\ Dp_k \\ Dp_{k+1} \end{bmatrix}}_{\text{boundary info}}$$

Specific Example: Cardinal Splines

Using same methods as with Hermite spline, from boundary conditions on previous slide we can get

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{M_{\text{Hermite}}} \begin{bmatrix} p_k \\ p_{k+1} \\ s(p_{k+1} - p_{k-1}) \\ s(p_{k+2} - p_k) \end{bmatrix}$$

where $s = (1 - t)/2$

The parameter t is called the <u>tension parameter</u>.

- Controls looseness versus tightness of curve

Specific Example: Cardinal Splines

We can express the boundary conditions as a matrix applied to the points p_{k-1}, p_k, p_{k+1}, and p_{k+2}:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ s(p_{k+1} - p_{k-1}) \\ s(p_{k+2} - p_k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

to get

Specific Example: Cardinal Splines

We can express the boundary conditions as a matrix applied to the points p_{k-1}, p_k, p_{k+1}, and p_{k+2}:

$$\begin{bmatrix} p_k \\ p_{k+1} \\ s(p_{k+1} - p_{k-1}) \\ s(p_{k+2} - p_k) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

to get

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

Specific Example: Cardinal Splines

Multiplying the interior matrices in:

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

we get the Cardinal matrix representation

Specific Example: Cardinal Splines

Combining the matrices in:

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & s & 0 \\ 0 & -s & 0 & s \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

we get the Cardinal matrix representation

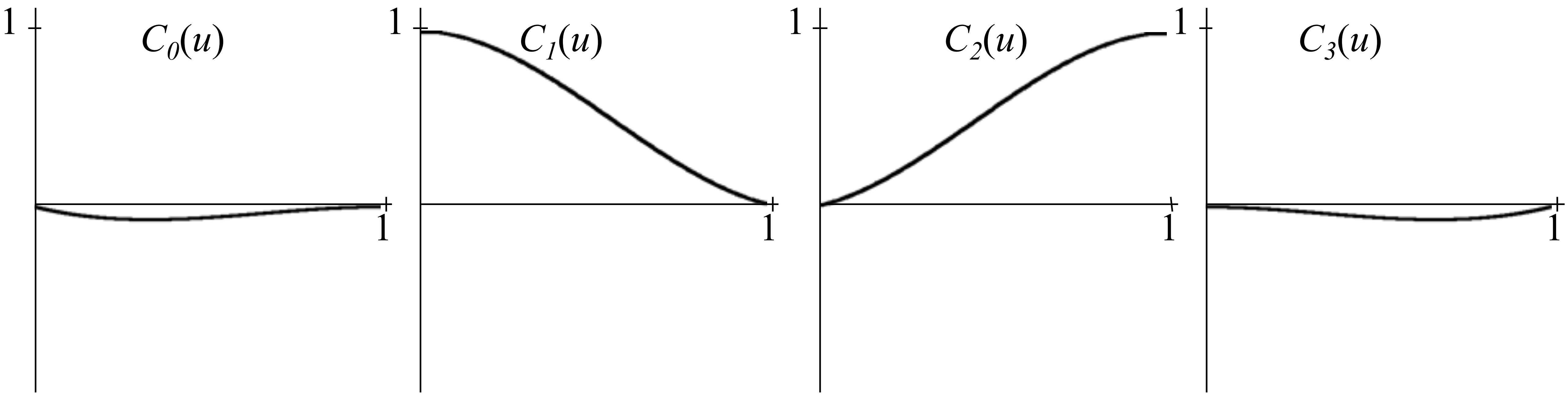
$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{M_{\text{Cardinal}}} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

Specific Example: Cardinal Splines

Setting:

$$\left.\begin{array}{l} \circ\ C_0(u) = -su^3+2su^2-su \\ \circ\ C_1(u) = (2-s)u^3+(s-3)u^2+1 \\ \circ\ C_2(u) = (s-2)u^3+(3-2s)u^2+su \\ \circ\ C_3(u) = su^3-su^2 \end{array}\right\} \text{Blending Functions}$$

For $s=0$:



$$P(u) = C_0(u)p_{k-1} + C_1(u)p_k + C_2(u)p_{k+1} + C_3(u)p_{k+2}$$

Specific Example: Cardinal Splines

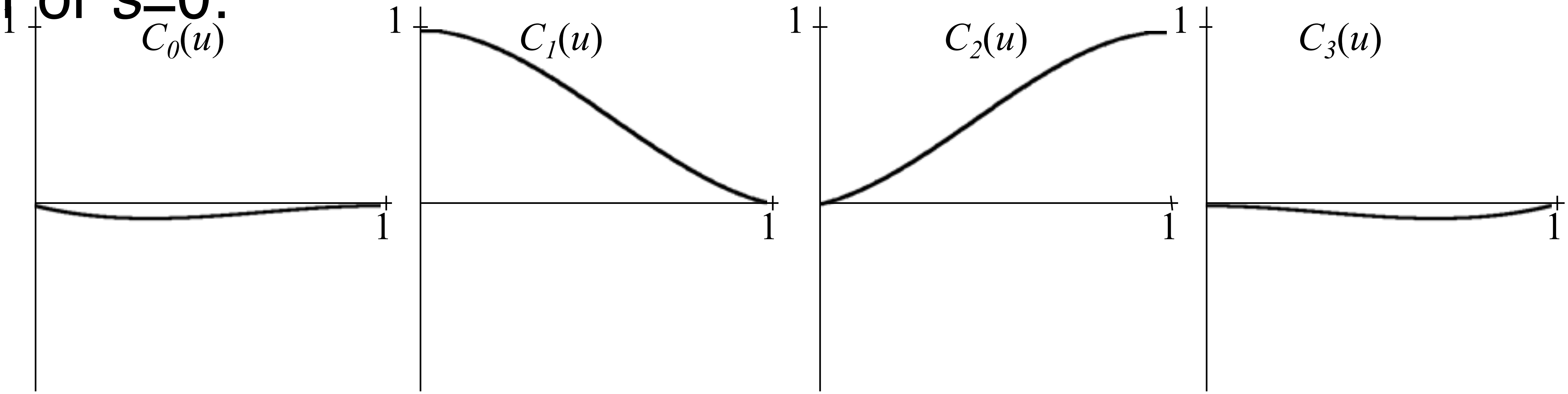
Setting:

- $C_0(u) = -su^3+2su^2-su$
- $C_1(u) = (2-s)u^3+(s-3)u^2+1$
- $C_2(u) = (s-2)u^3+(3-2s)u^2+su$
- $C_3(u) = su^3-su^2$

<u>Properties</u>:

- $C_0(u)+C_1(u)+C_2(u)+C_3(u)=1$
- $C_j(u)=C_{3-j}(1-u)$
- $C_0(1)=C_3(0)=0$

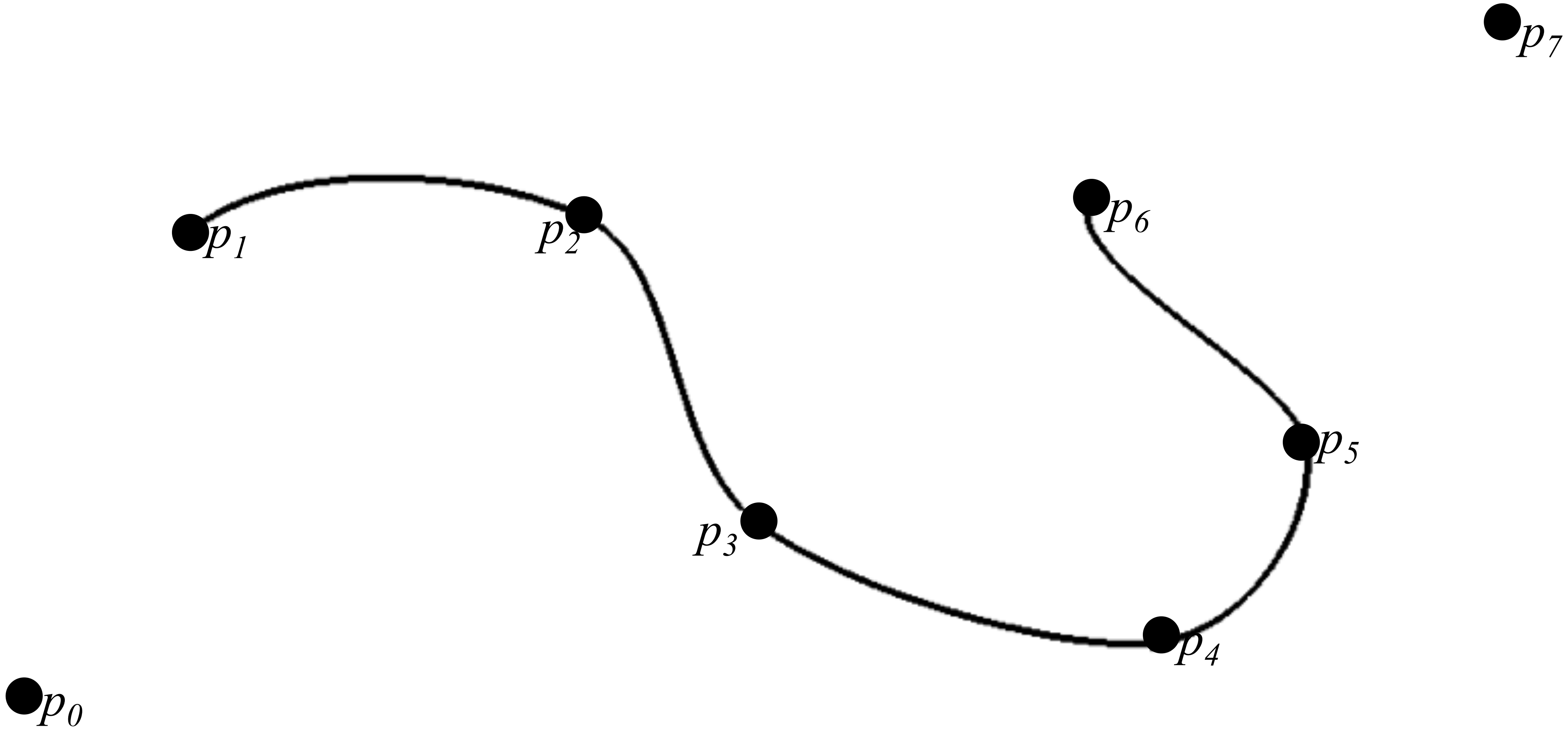
For s=0:



$$P(u) = C_0(u)p_{k-1} + C_1(u)p_k + C_2(u)p_{k+1} + C_3(u)p_{k+2}$$

Specific Example: Cardinal Splines

- Interpolating piecewise *cubic* polynomial
- Specified with four control points
- Iteratively construct the curve between middle two points using adjacent points to define tangents



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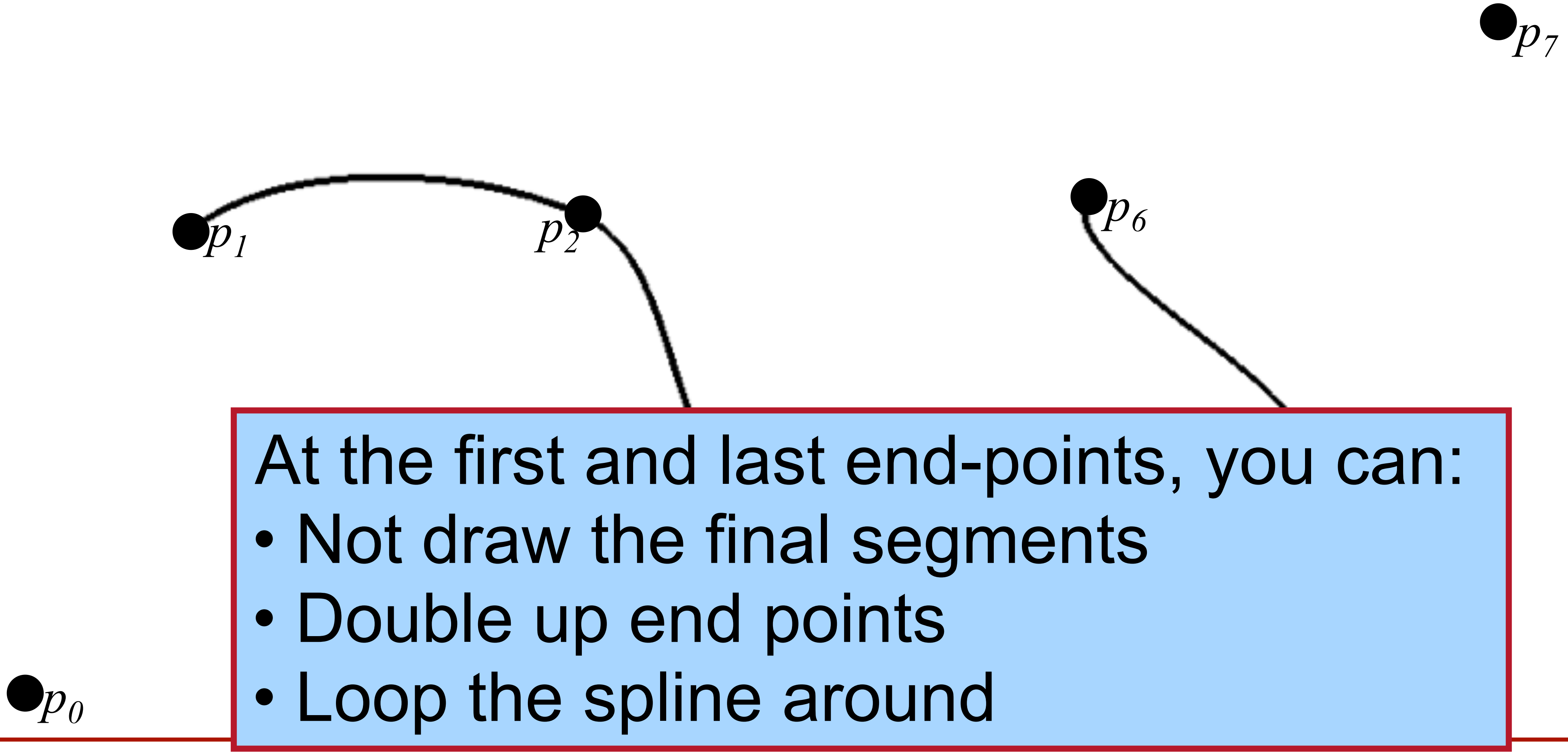

At the first and last end-points, you can:

- Not draw the final segments
- Double up end points
- Loop the spline around

Overview

- What is a Spline?
- Specific Examples:
  - Hermite Splines
  - Cardinal Splines
  - Uniform Cubic B-Splines
- Comparing Cardinal Splines to Uniform Cubic B-Splines

Specific Example: Uniform Cubic B-Splines

- Approximating piecewise *cubic* polynomial
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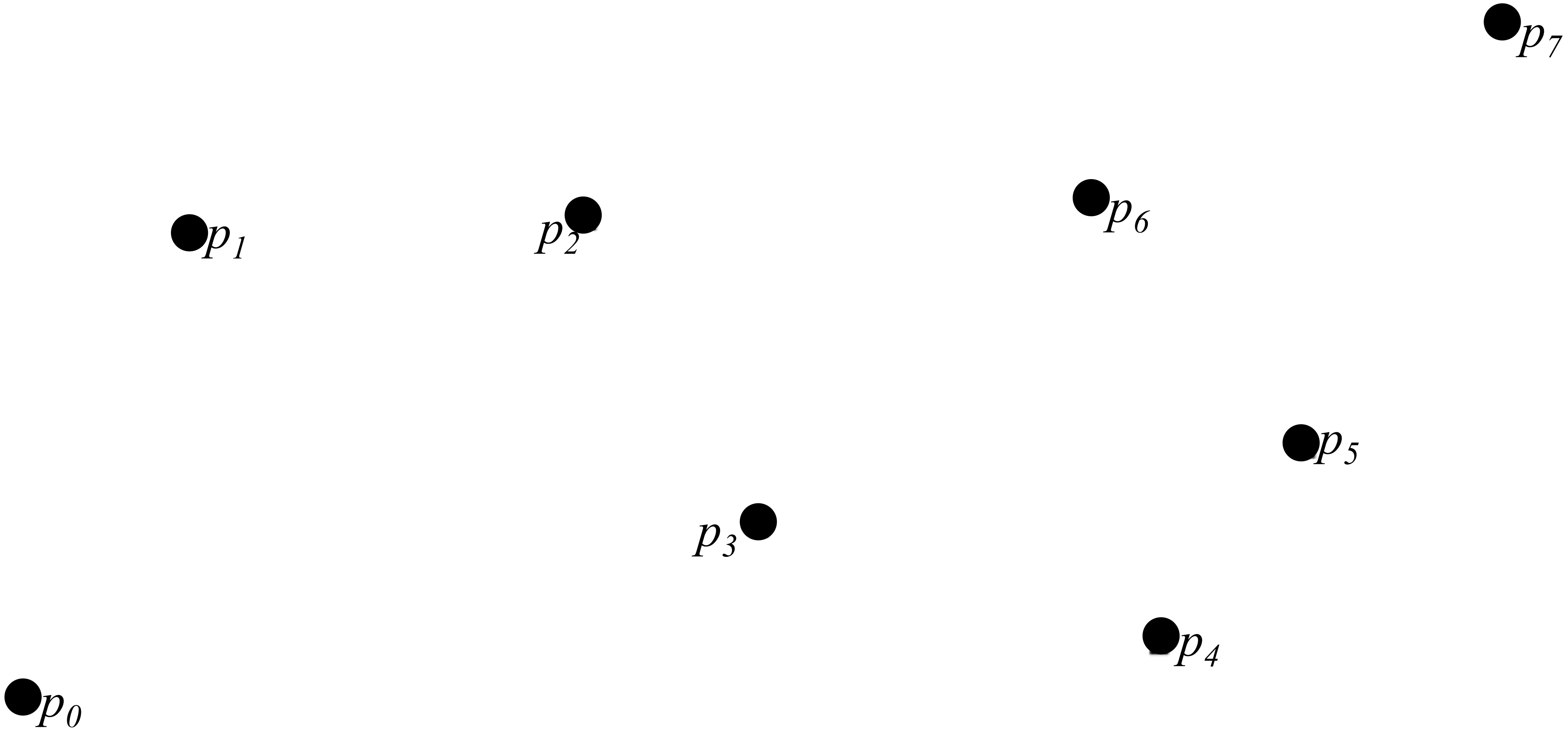

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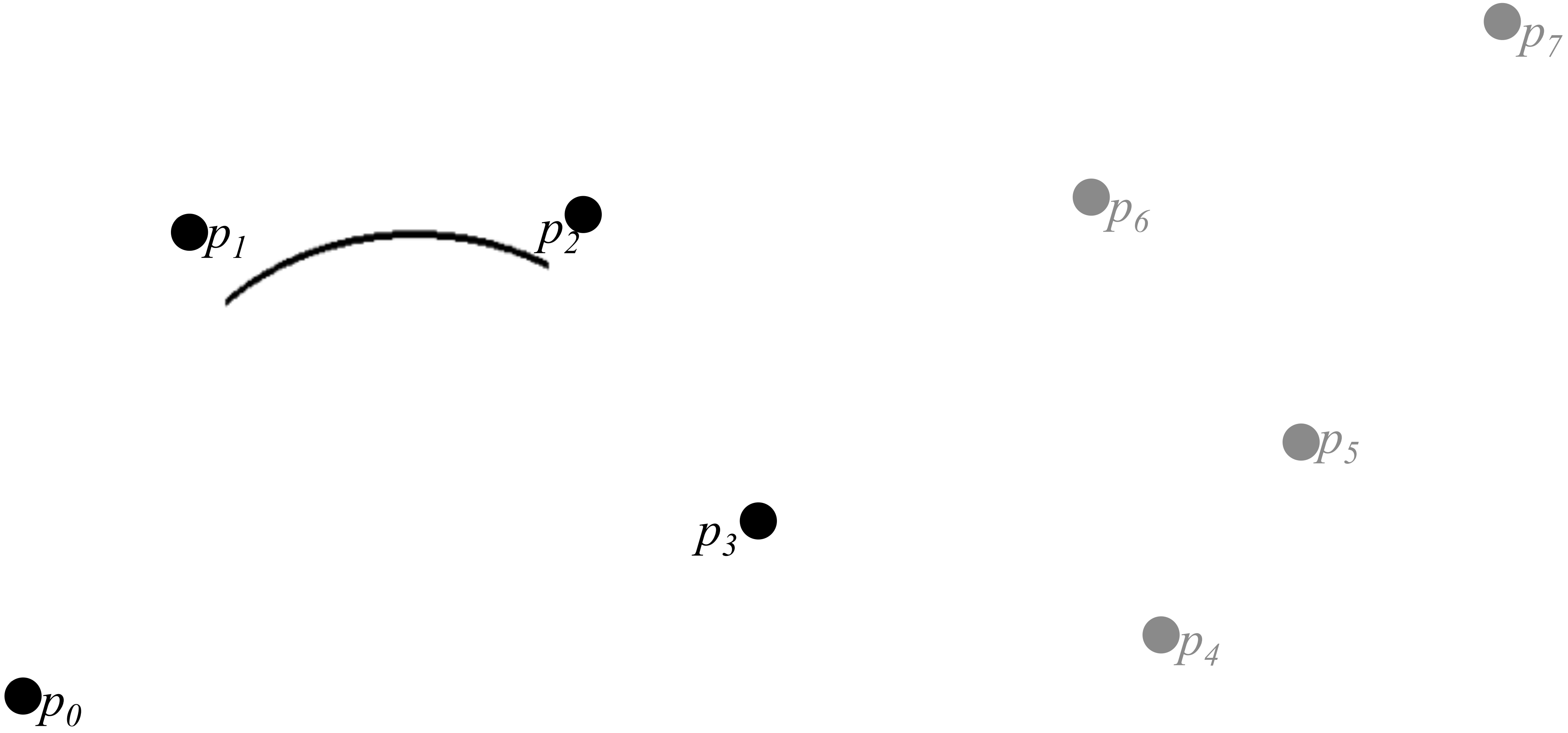

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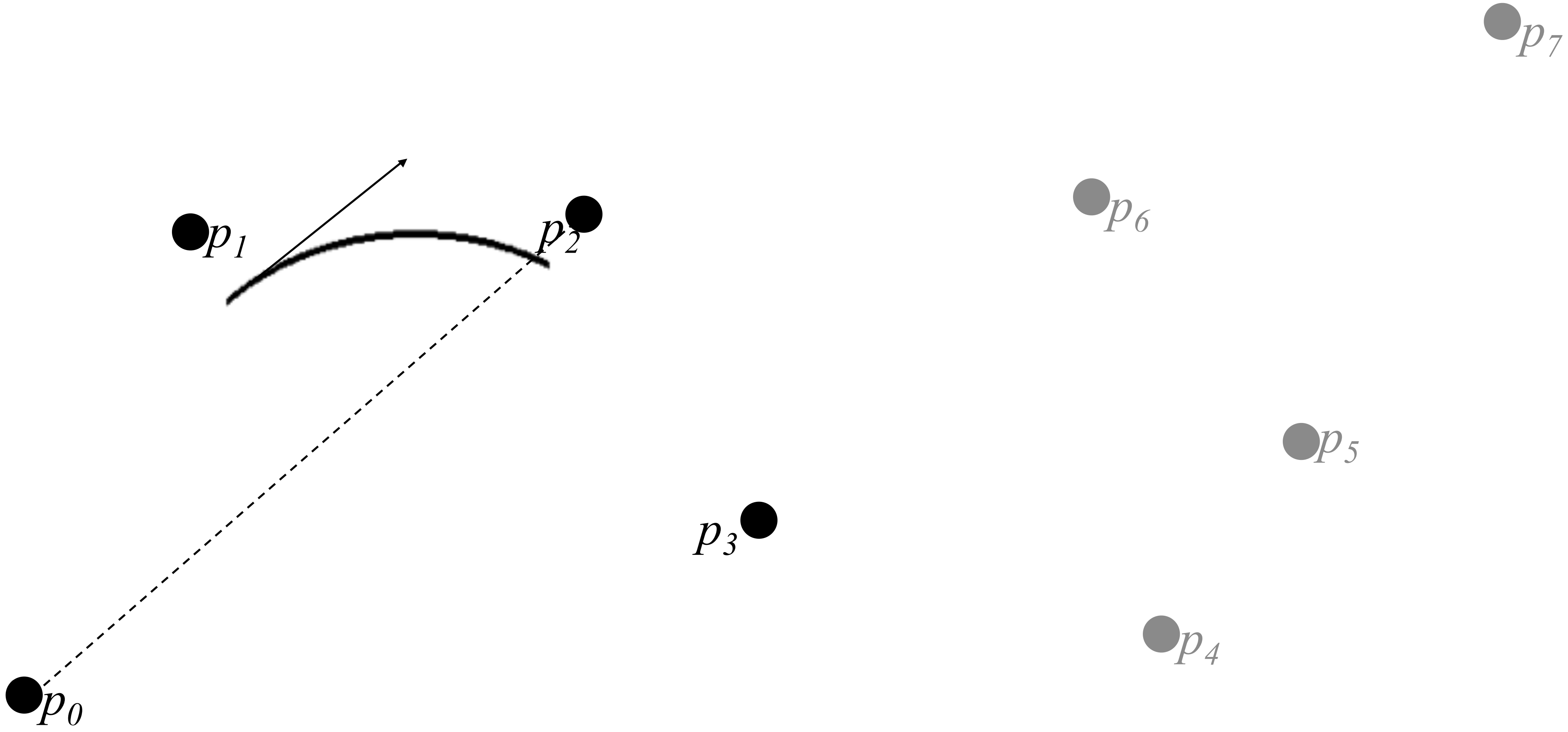

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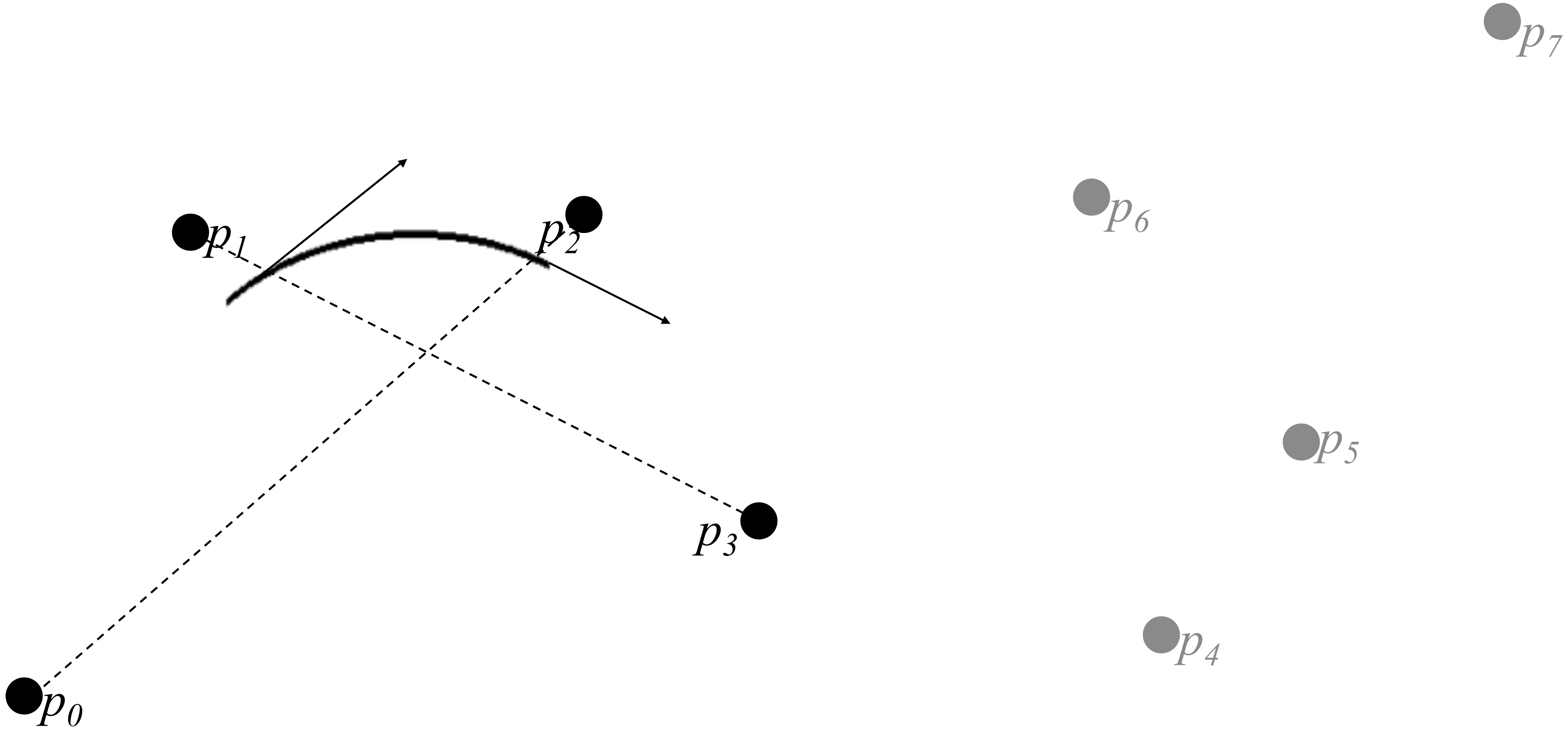

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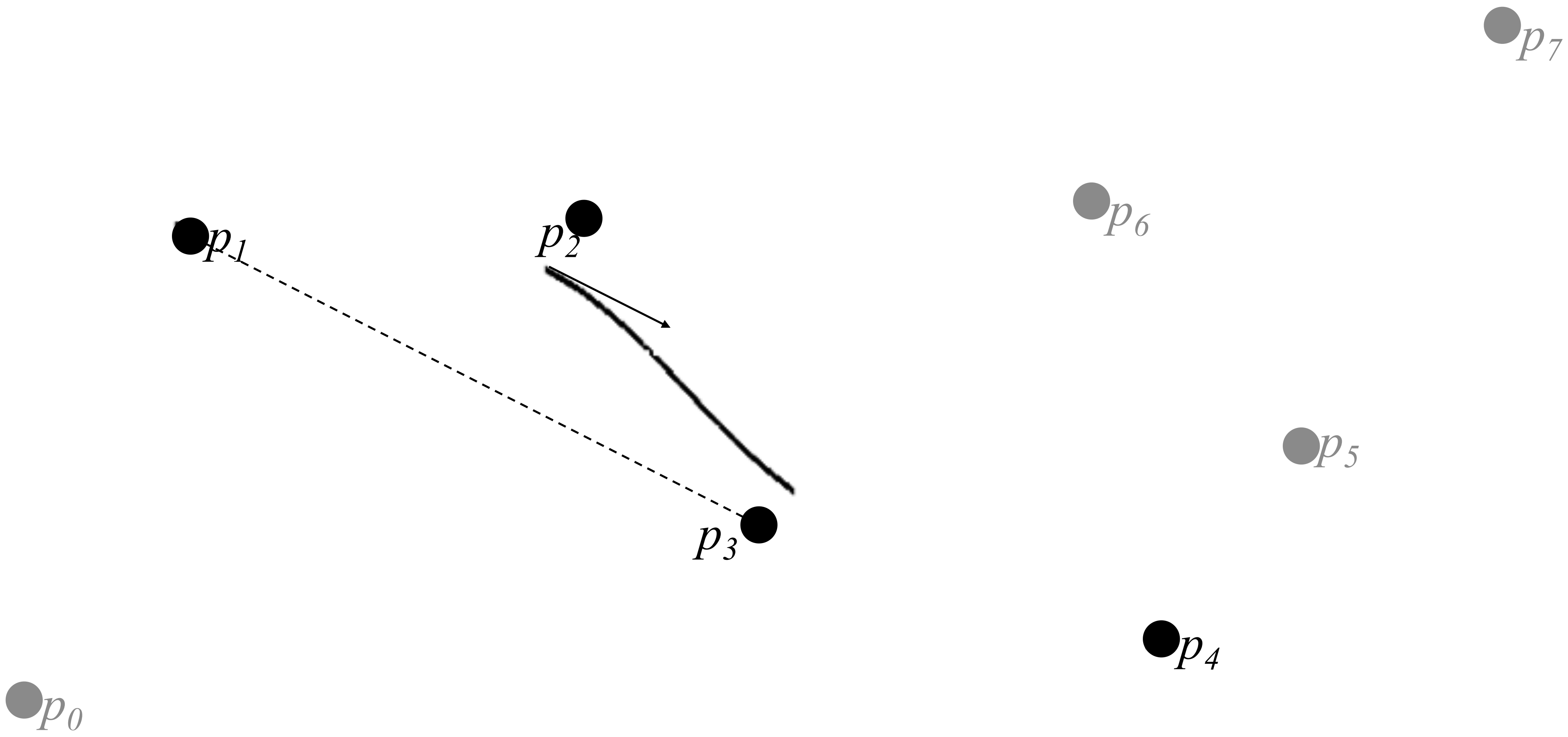

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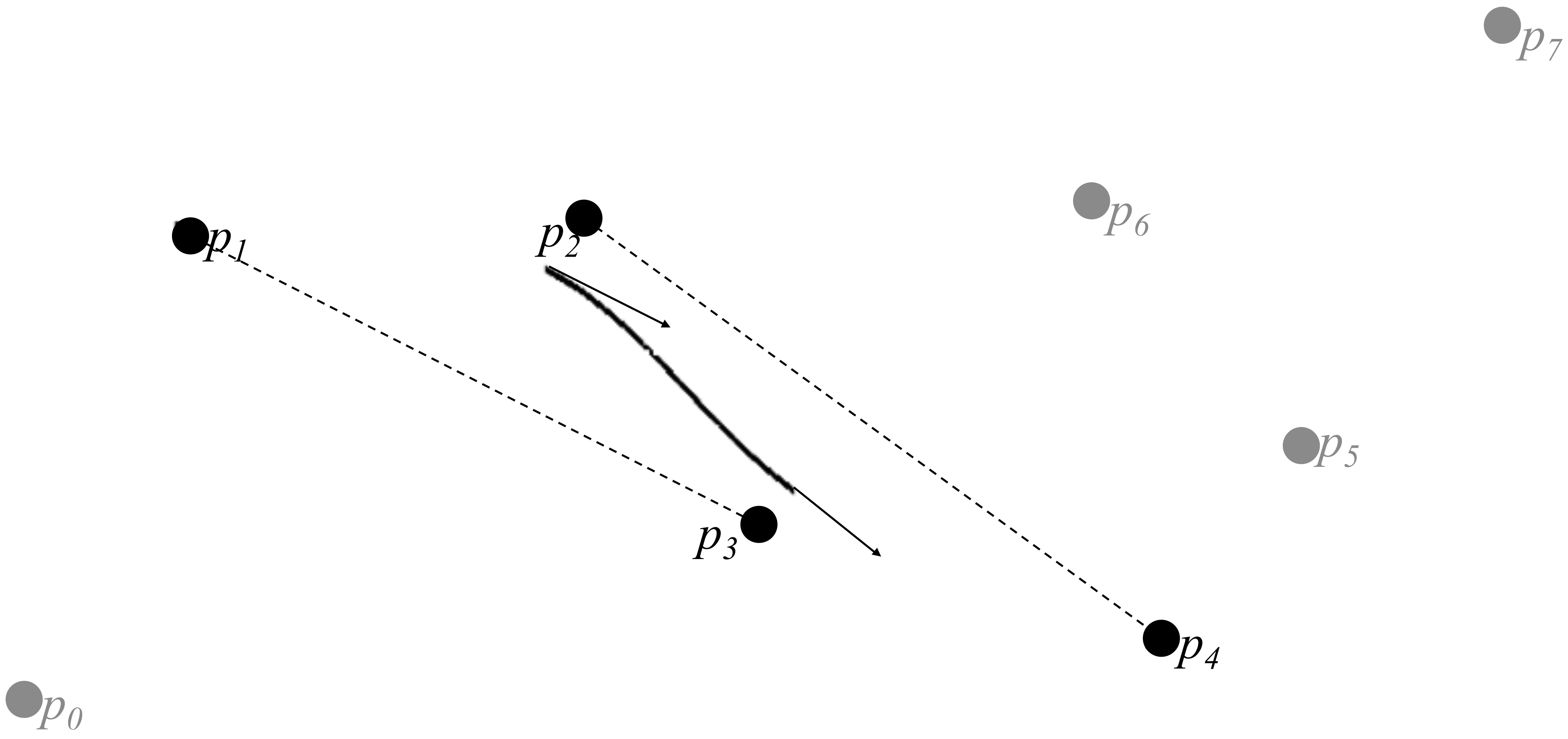

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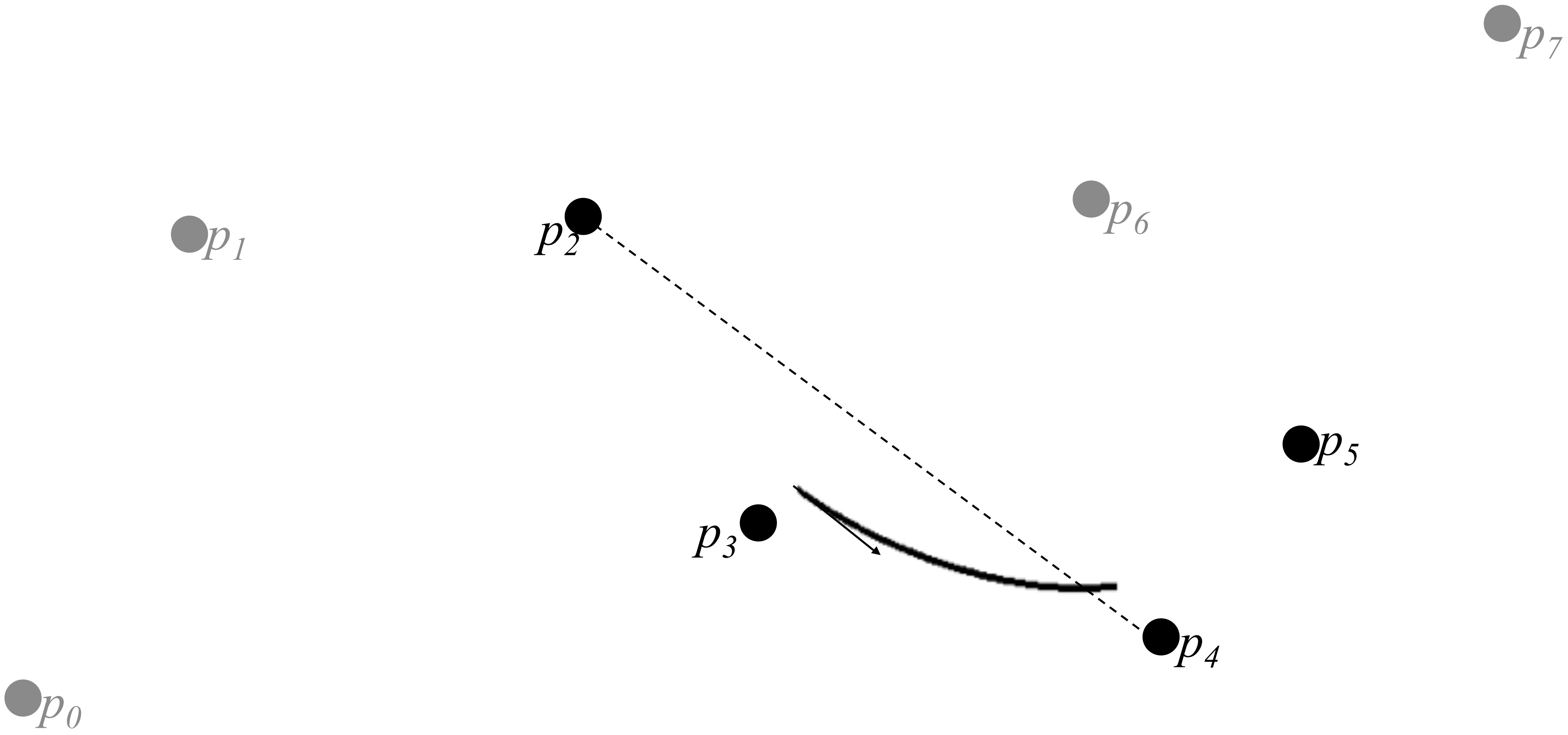

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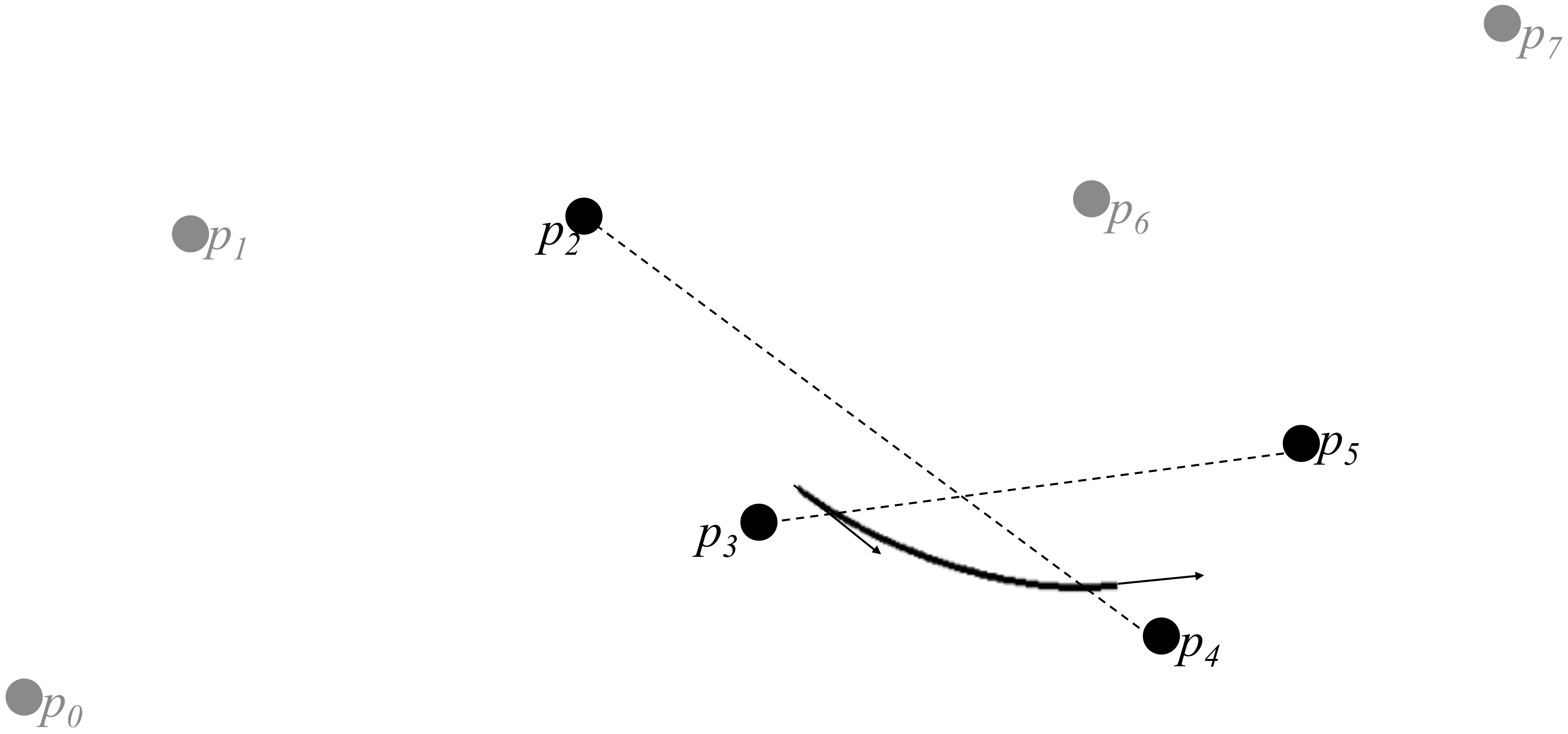

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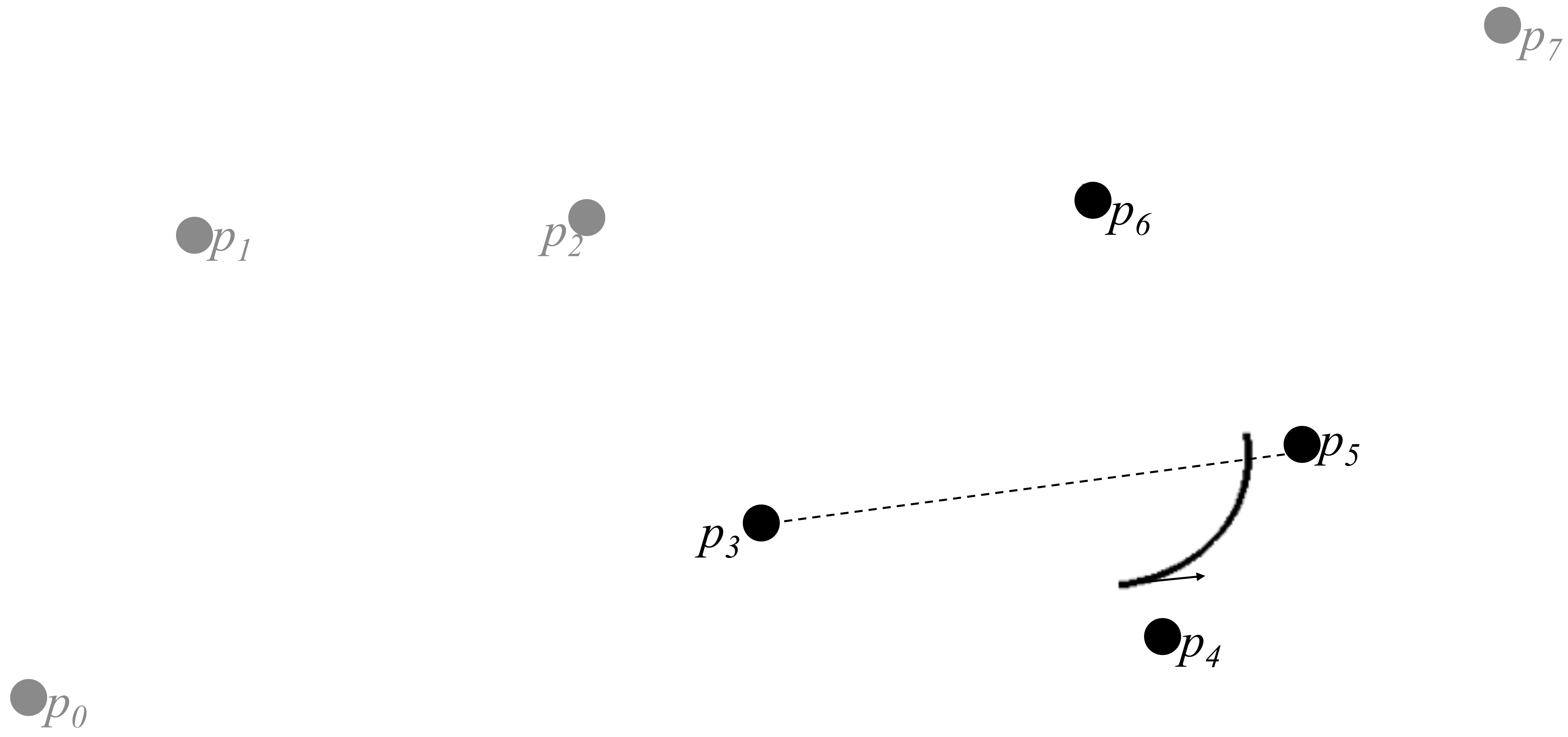

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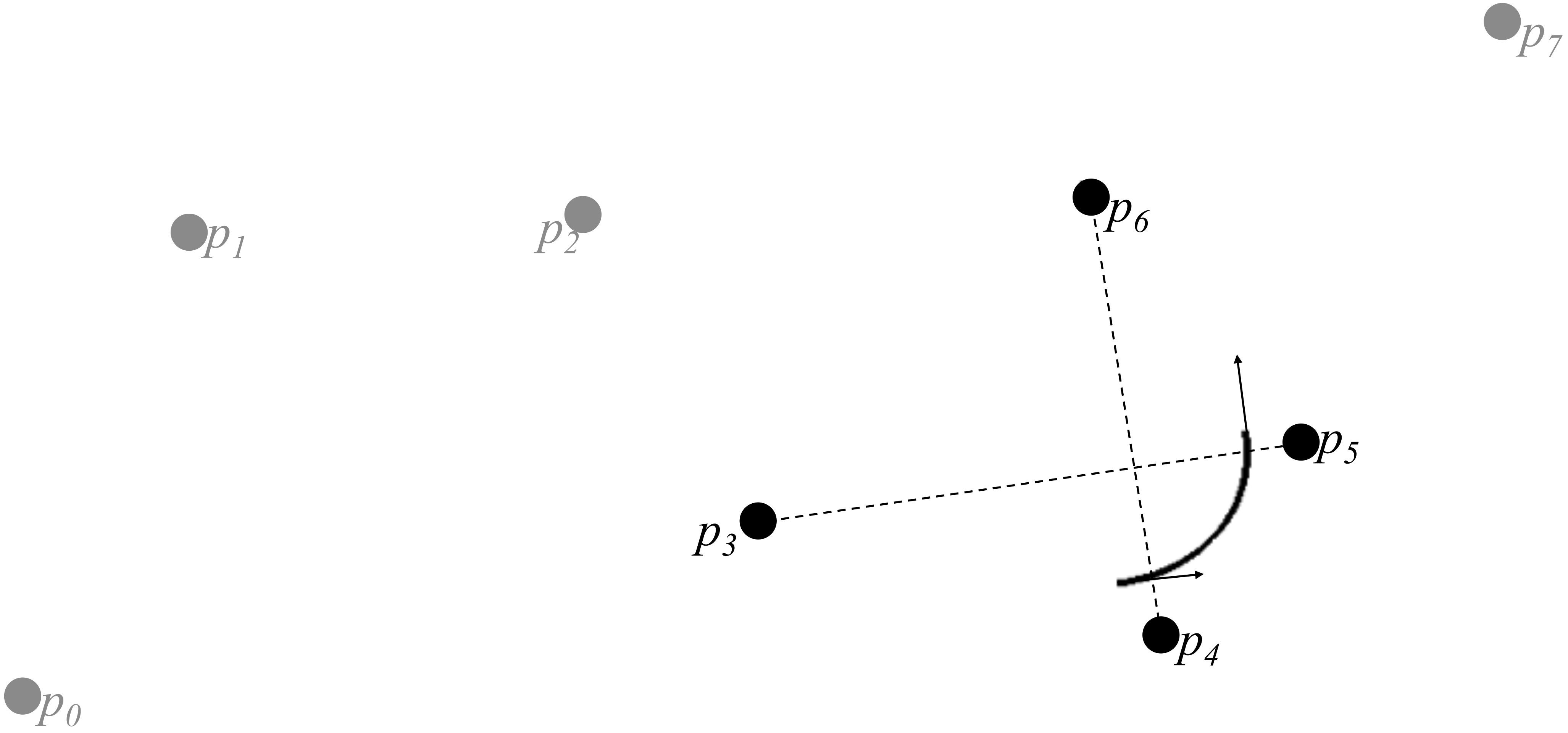

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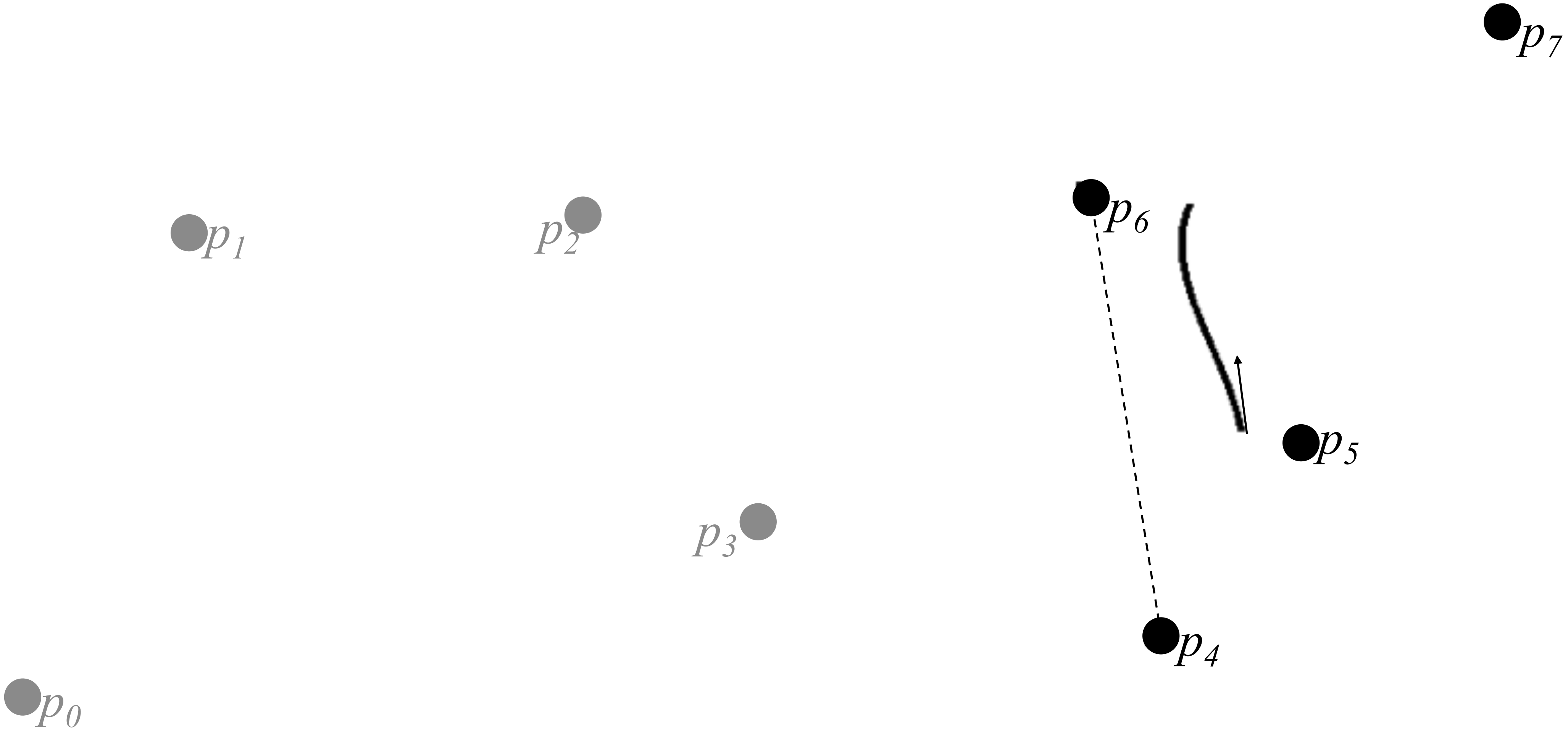

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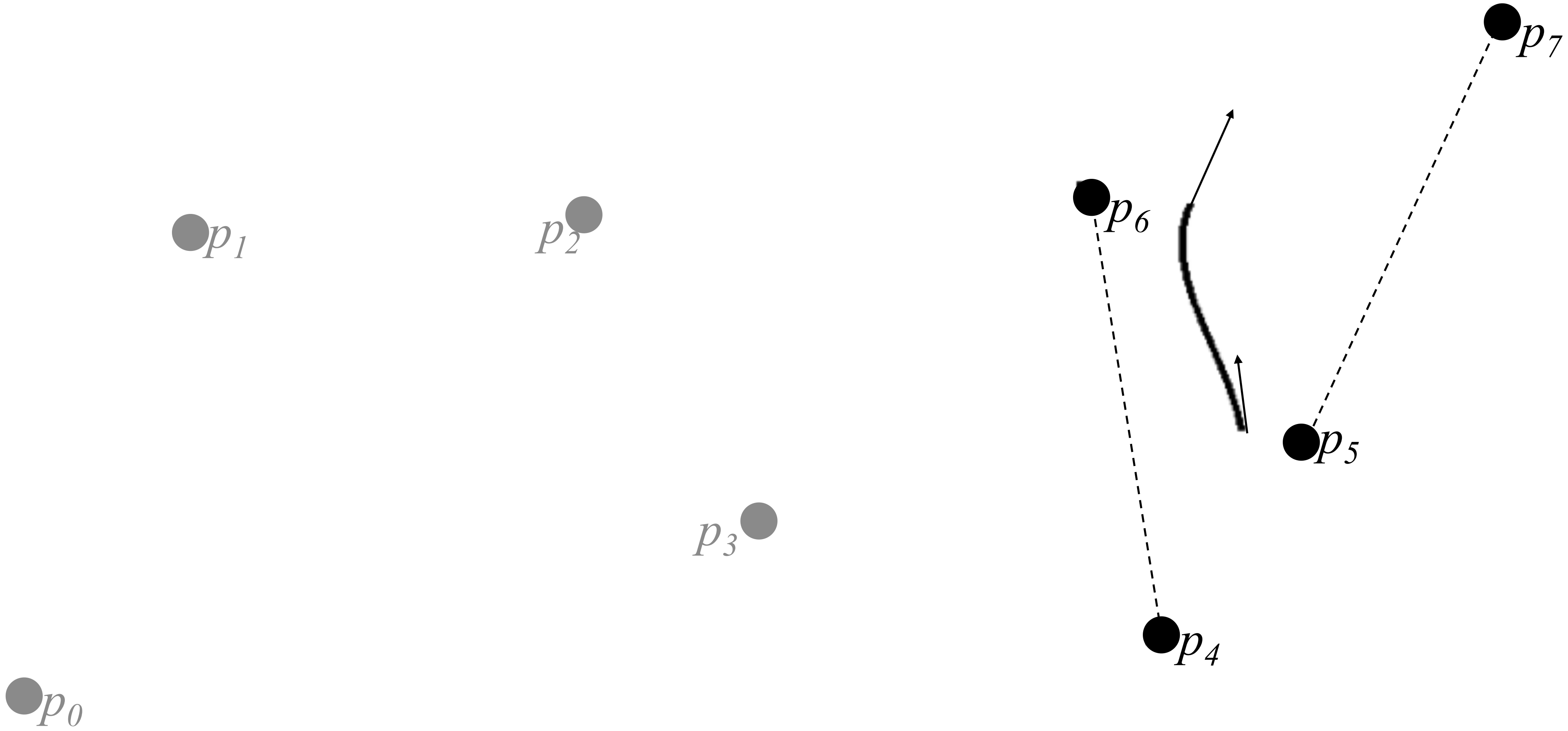

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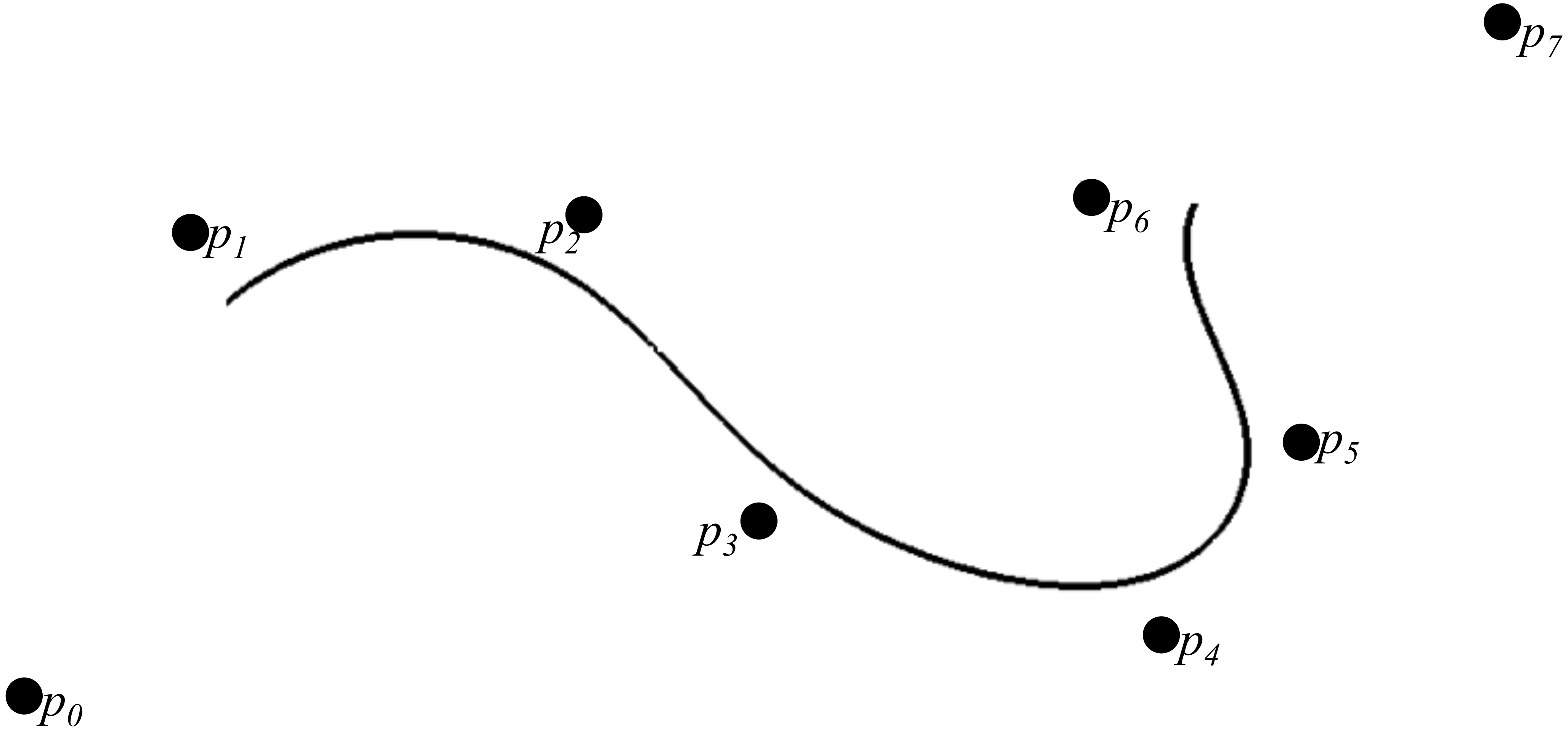
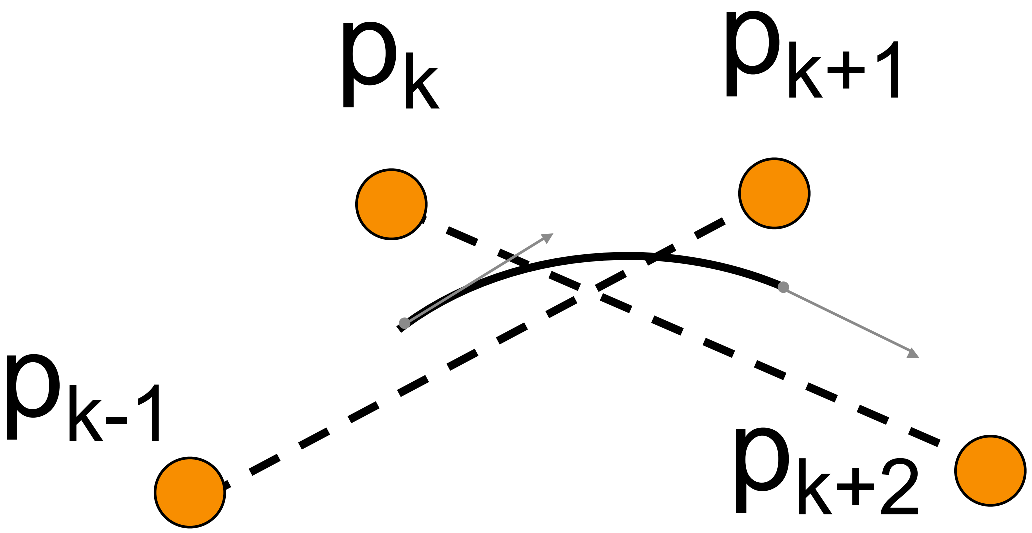


Specific Example: Uniform Cubic B-Splines

- Let $P_k(u)=(P_{k,X}(u),P_{k,Y}(u))$ with $0 \leq u \leq 1$ be a parametric cubic point function for the curve section around the control points p_k and p_{k+1}
- Boundary conditions are:
  - $P(0) = 1/6(p_{k-1}+4p_k+p_{k+1})$
  - $P(1) = 1/6(p_k+4p_{k+1}+p_{k+2})$
  - $P'(0) = 1/2(1-t)(p_{k+1}-p_{k-1})$
  - $P'(1) = 1/2(1-t)(p_{k+2}-p_k)$



- Solve for the coefficients of the polynomials $P_{k,X}(u)$ and $P_{k,Y}(u)$ that satisfy the boundary condition

Specific Example: Uniform Cubic B-Splines

Using same methods as with Hermite spline, from boundary conditions on previous slide we can get

$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{M_{\text{Hermite}}} \frac{1}{6} \begin{bmatrix} p_{k-1} + 4p_k + p_{k+1} \\ p_k + 4p_{k+1} + p_{k+2} \\ 3p_{k+1} - 3p_{k-1} \\ 3p_{k+2} - 3p_k \end{bmatrix}$$

Specific Example: Uniform Cubic B-Splines

We can express the boundary conditions as a matrix applied to the points p_{k-1}, p_k, p_{k+1}, and p_{k+2}:

$$\begin{bmatrix} p_{k-1} + 4p_k + p_{k+1} \\ p_k + 4p_{k+1} + p_{k+2} \\ 3p_{k+1} - 3p_{k-1} \\ 3p_{k+2} - 3p_k \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

to get

$$P(u) = \frac{1}{6} \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

Specific Example: Uniform Cubic B-Splines

Multiplying the interior matrices in:

$$P(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

we get the cubic B-spline matrix representation

Specific Example: Uniform Cubic B-Splines

Combining the matrices in:

$$P(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \end{bmatrix}\begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

we get the cubic B-spline matrix representation

$$P(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}\begin{bmatrix} p_{k-1} \\ p_k \\ p_{k+1} \\ p_{k+2} \end{bmatrix}$$

$$\uparrow$$

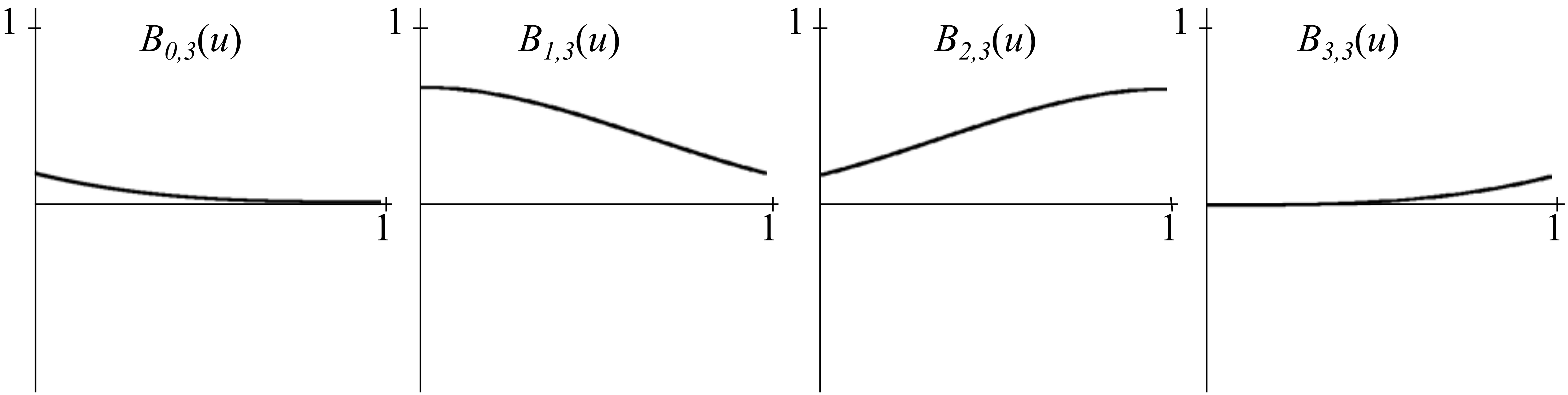
$$\boldsymbol{M}_{\mathbf{BSpline}}$$

Specific Example: Uniform Cubic B-Splines

Setting:

- $B_{0,3}(u)= 1/6(1-u)^3$
- $B_{1,3}(u)= 1/6(3u^3-6u^2+4)$
- $B_{2,3}(u)= 1/6(-3u^3+3u^2+3u+1)$
- $B_{3,3}(u)= 1/6(u^3)$

Blending Functions



$$P(u) = B_{0,3}(u)p_{k-1} + B_{1,3}(u)p_k + B_{2,3}(u)p_{k+1} + B_{3,3}(u)p_{k+2}$$

Specific Example: Uniform Cubic B-Splines

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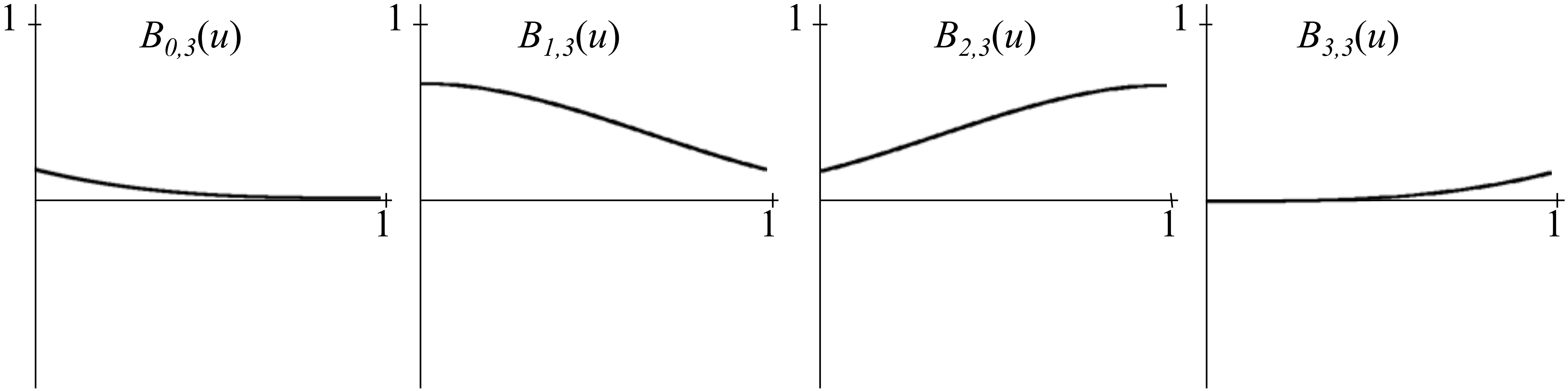
Properties:

- $B_{0,3}(u)+B_{1,3}(u)+B_{2,3}(u)+B_{3,3}(u)=1$
- $B_j(u)=B_{3-j}(1-u)$
- $B_{0,3}(1)=B_{3,3}(0)=0$
- $B_{j,3}(u)\geq 0$



$$P(u) = B_{0,3}(u)p_{k-1} + B_{1,3}(u)p_k + B_{2,3}(u)p_{k+1} + B_{3,3}(u)p_{k+2}$$

Specific Example: Uniform Cubic B-Splines

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- Specified with four control points
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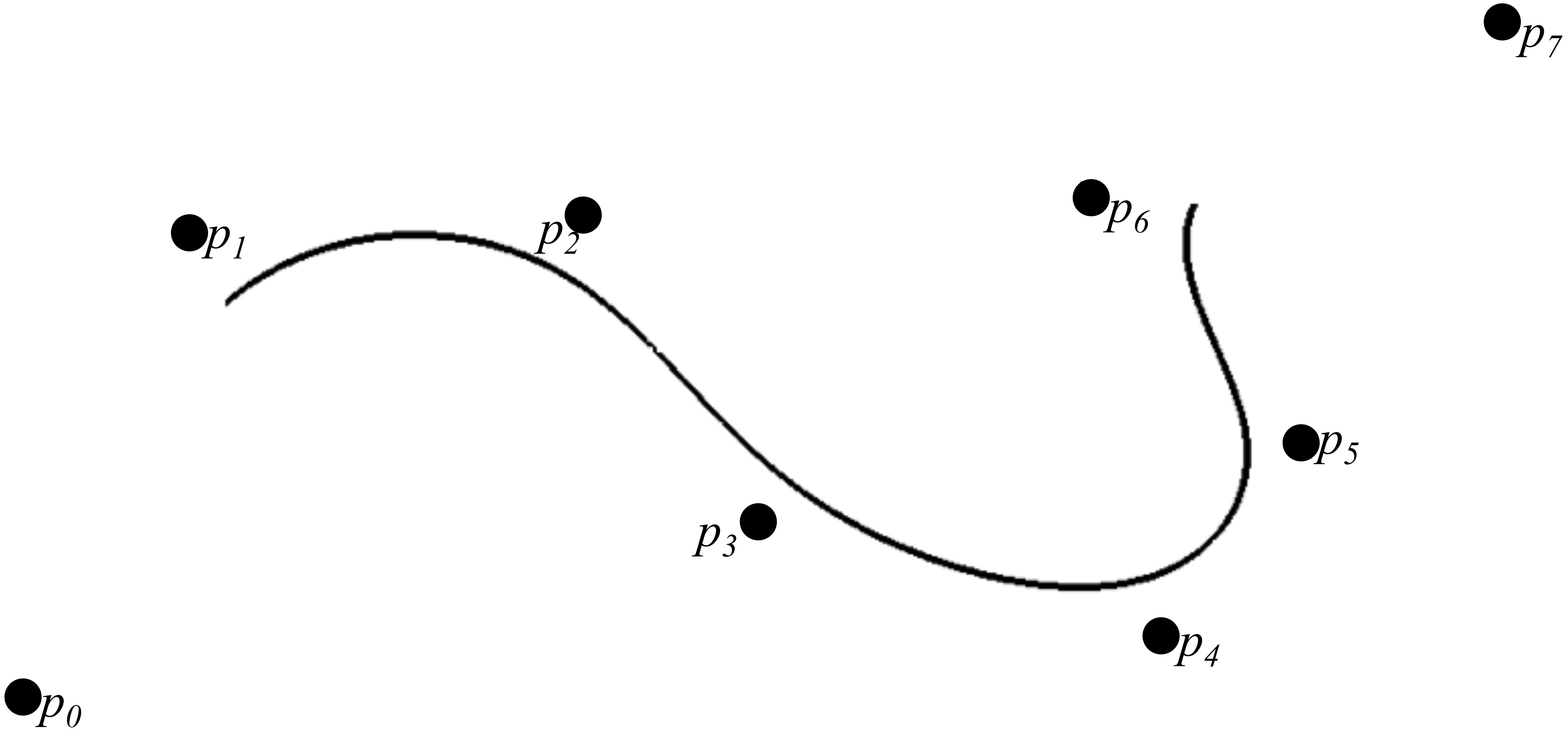

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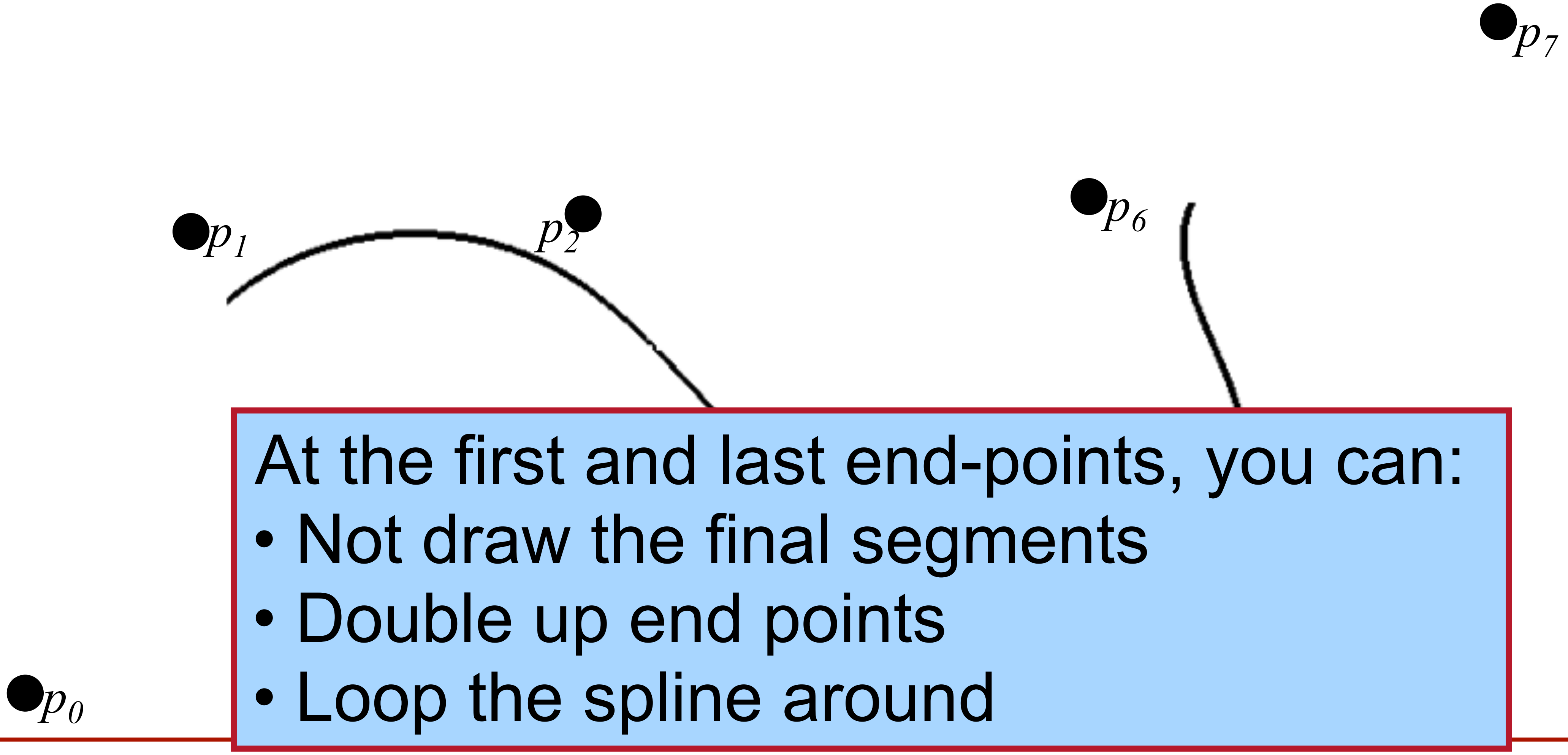

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- Not draw the final segments
- Double up end points
- Loop the spline around

Overview

- What is a Spline?
- Specific Examples:
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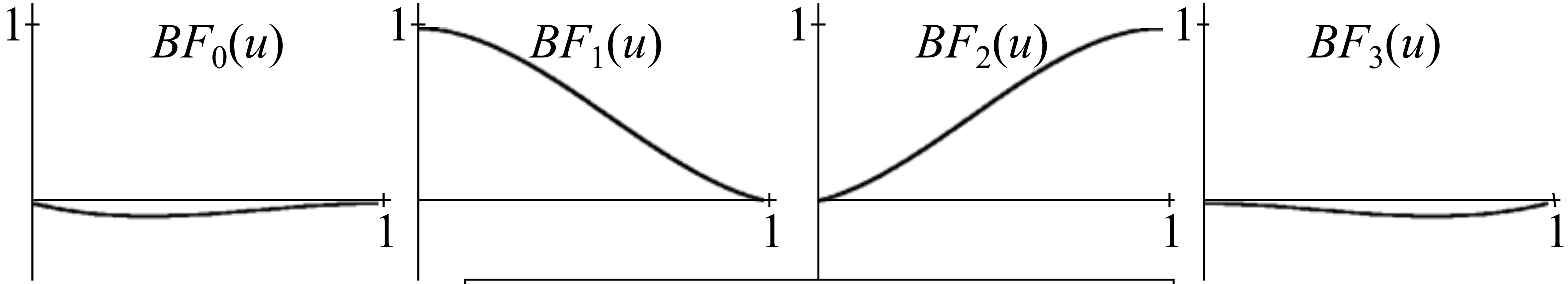
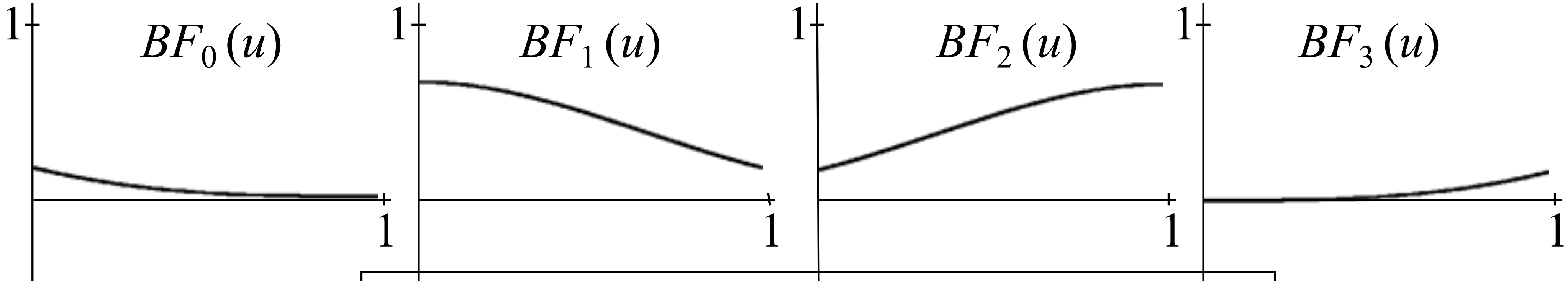
Blending Functions

Blending functions provide a way for expressing the functions $P_k(u)$ as a weighted sum of the four control points p_{k-1}, p_k, p_{k+1}, and p_{k+2}:



Cardinal Blending Functions (t=0)



Uniform Cubic B-Spline Blending Functions

$$P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}$$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$, for all $0 \leq u \leq 1$.

Blending Functions

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- Translation Commutativity:
  - $BF_0(u)+BF_1(u)+BF_2(u)+BF_3(u)=1$, for all $0 \le u \le 1$.

    If we translate all the control points by the same vector *q*, the position of the new point at the value *u* will just be the position of the old value at *u*, translated by *q*:

Blending Functions

Properties:

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    If we translate all the control points by the same vector *q*, the position of the new point at the value *u* will just be the position of the old value at *u*, translated by *q*:

$$Q_k(u) = BF_0(u)(q + p_{k-1}) + BF_1(u)(q + p_k) + BF_2(u)(q + p_{k+1}) + BF_3(u)(q + p_{k+2})$$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u)+BF_1(u)+BF_2(u)+BF_3(u)=1$, for all $0 \leq u \leq 1$.

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$$
\begin{aligned}
Q_k(u) &= BF_0(u)\left(q+p_{k-1}\right)+BF_1(u)\left(q+p_k\right)+BF_2(u)\left(q+p_{k+1}\right)+BF_3(u)\left(q+p_{k+2}\right) \\
&= \left(BF_0(u)+BF_1(u)+BF_1(u)+BF_1(u)\right)q+P_k(u)
\end{aligned}
$$

Blending Functions

Properties:

- Translation Commutativity:
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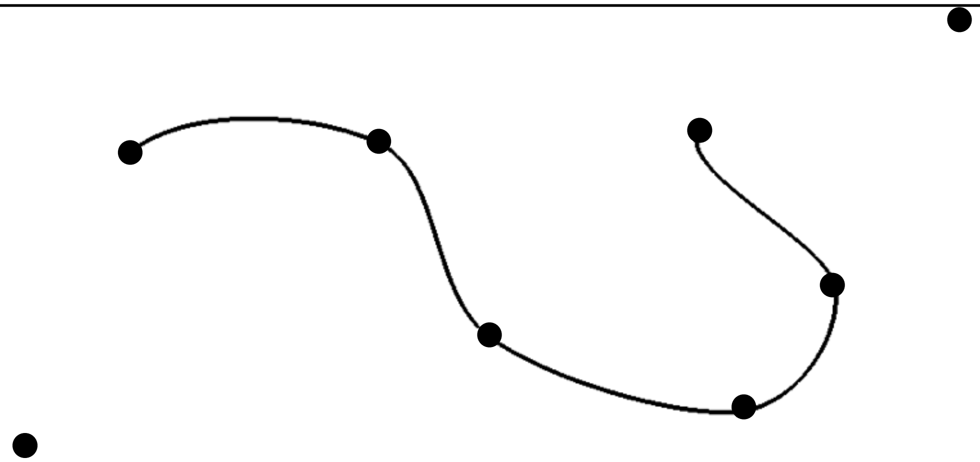
$$
\begin{aligned}
Q_k(u) &= BF_0(u)\left(q + p_{k-1}\right) + BF_1(u)\left(q + p_k\right) + BF_2(u)\left(q + p_{k+1}\right) + BF_3(u)\left(q + p_{k+2}\right) \\
&= \left(BF_0(u) + BF_1(u) + BF_1(u) + BF_1(u)\right)q + P_k(u) \\
&= q + P_k(u)
\end{aligned}
$$

Comparison: Cardinal vs. Cubic B

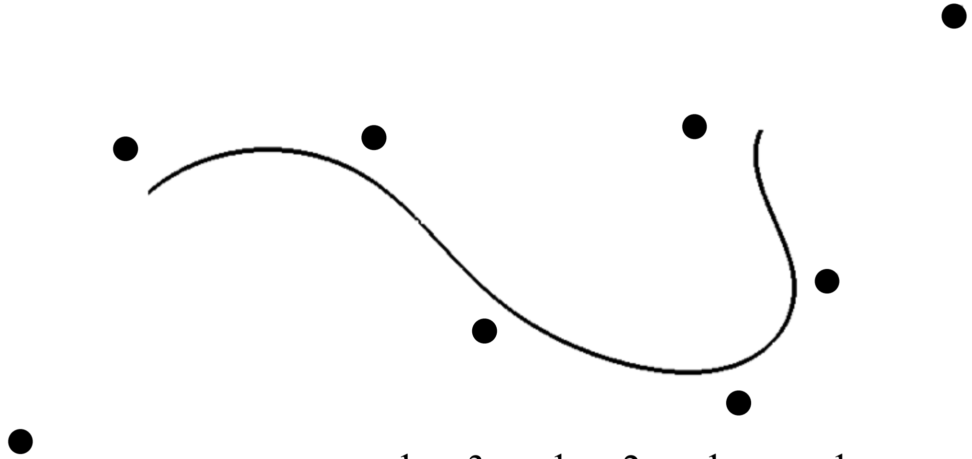
Cardinal Splines (t=0)



$$BF_0(u) = -\tfrac{1}{2}u^3 + \ u^2 - \tfrac{1}{2}u$$
$$BF_1(u) = \ \tfrac{3}{2}u^3 - \tfrac{5}{2}u^2 + \quad +1$$
$$BF_2(u) = -\tfrac{3}{2}u^3 + 2u^2 + \tfrac{1}{2}u$$
$$BF_3(u) = \ \tfrac{1}{2}u^3 - \tfrac{1}{2}u^2$$

$$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$$

Cubic B-Splines



$$BF_0(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_1(u) = \ \tfrac{1}{2}u^3 - \ u^2 + \quad + \tfrac{2}{3}$$
$$BF_2(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_3(u) = \ \tfrac{1}{6}u^3$$

$$BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$$

$$\boxed{P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}}$$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u)+ BF_1(u)+ BF_2(u)+ BF_3(u)=1$, for all $0 \leq u \leq 1$.
- Continuity:
  - $BF_0(1)=BF_3(0)=0$
  - $BF_1(1)=BF_0(0)$
  - $BF_2(1)=BF_1(0)$
  - $BF_3(1)=BF_2(0)$

Blending Functions

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We need to have the curve $P_{k+1}(u)$ begin where the curve $P_k(u)$ ended:

$$0 = P_{k+1}(0) - P_k(1)$$

Blending Functions

Properties:

- Translation Commutativity:
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Since this equation has to hold true regardless of the values of p_k, the conditions on the left have to be true

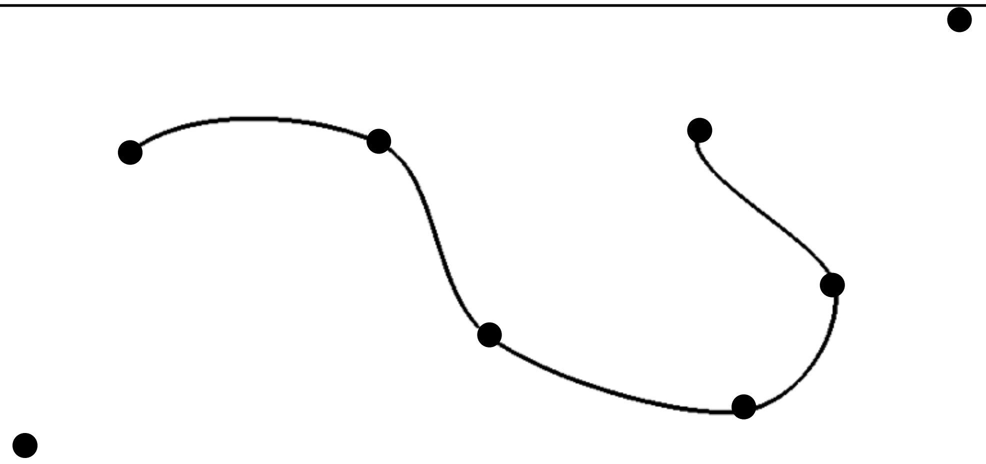
We need to have the curve $P_{k+1}(u)$ begin where the curve $P_k(u)$ ended:

$$0 = \mathrm{P}_{k+1}(0) - \mathrm{P}_k(1)$$

$$\begin{aligned} 0 = & \left(\qquad\qquad - BF_0(1) \right) p_{k-1} \\ & + \left(BF_0(0) - BF_1(1) \right) p_k \\ & + \left(BF_1(0) - BF_2(1) \right) p_{k+1} \\ & + \left(BF_2(0) - BF_3(1) \right) p_{k+2} \\ & + \left(BF_3(0) \qquad\qquad \right) p_{k+3} \end{aligned}$$

Comparison: Cardinal vs. Cubic B

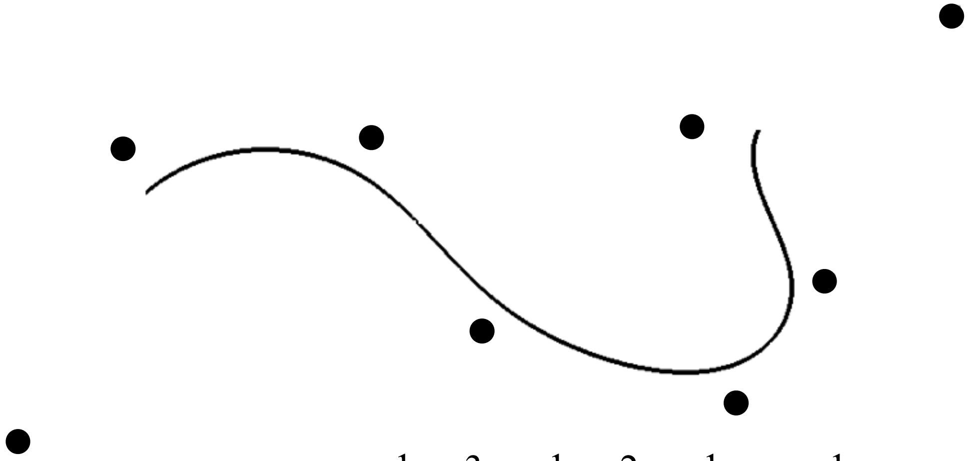
Cardinal Splines (t=0)



$$
\begin{aligned}
BF_0(u) &= -\tfrac{1}{2}u^3 + \ u^2 - \tfrac{1}{2}u \\
BF_1(u) &= \ \ \tfrac{3}{2}u^3 - \tfrac{5}{2}u^2 + \quad + 1 \\
BF_2(u) &= -\tfrac{3}{2}u^3 + 2u^2 + \tfrac{1}{2}u \\
BF_3(u) &= \ \ \tfrac{1}{2}u^3 - \tfrac{1}{2}u^2
\end{aligned}
$$

$$
\begin{array}{lcl}
BF_0(0) = 0 & \searrow & BF_0(1) = 0 \\
BF_1(0) = 1 & \searrow & BF_1(1) = 0 \\
BF_2(0) = 0 & \searrow & BF_2(1) = 1 \\
BF_3(0) = 0 & & BF_3(1) = 0
\end{array}
$$

Cubic B-Splines



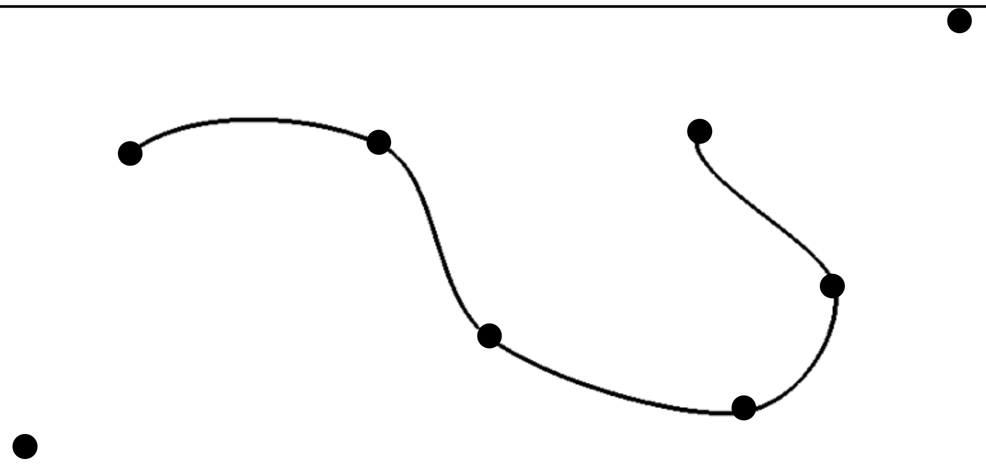
$$
\begin{aligned}
BF_0(u) &= -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6} \\
BF_1(u) &= \ \ \tfrac{1}{2}u^3 - \ u^2 + \quad + \tfrac{2}{3} \\
BF_2(u) &= -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6} \\
BF_3(u) &= \ \ \tfrac{1}{6}u^3
\end{aligned}
$$

$$
\begin{array}{lcl}
BF_0(0) = \tfrac{1}{6} & \searrow & BF_0(1) = 0 \\
BF_1(0) = \tfrac{2}{3} & \searrow & BF_1(1) = \tfrac{1}{6} \\
BF_2(0) = \tfrac{1}{6} & \searrow & BF_2(1) = \tfrac{2}{3} \\
BF_3(0) = 0 & & BF_3(1) = \tfrac{1}{6}
\end{array}
$$

$$
\boxed{P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}}
$$

Comparison: Cardinal vs. Cubic B

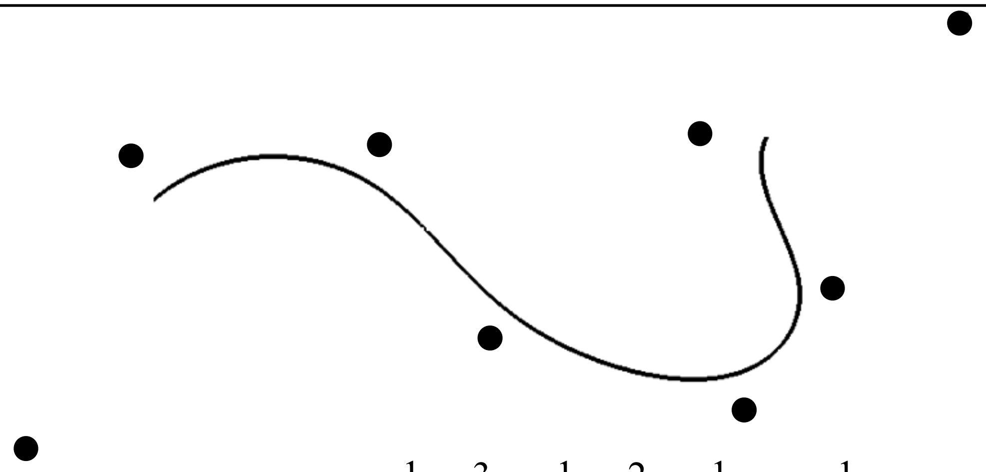
Cardinal Splines (t=0)



$$BF_0(u) = -\tfrac{1}{2}u^3 + \ u^2 - \tfrac{1}{2}u$$
$$BF_1(u) = \ \tfrac{3}{2}u^3 - \tfrac{5}{2}u^2 + \quad +1$$
$$BF_2(u) = -\tfrac{3}{2}u^3 + 2u^2 + \tfrac{1}{2}u$$
$$BF_3(u) = \ \tfrac{1}{2}u^3 - \tfrac{1}{2}u^2$$

$$\begin{tikzcd}
BF_0'(0) = -\tfrac{1}{2} \arrow[dr] & BF_0'(1) = 0 \\
BF_1'(0) = 0 \arrow[dr] & BF_1'(1) = -\tfrac{1}{2} \\
BF_2'(0) = \tfrac{1}{2} \arrow[dr] & BF_2'(1) = 0 \\
BF_3'(0) = 0 & BF_3'(1) = \tfrac{1}{2}
\end{tikzcd}$$

Cubic B-Splines



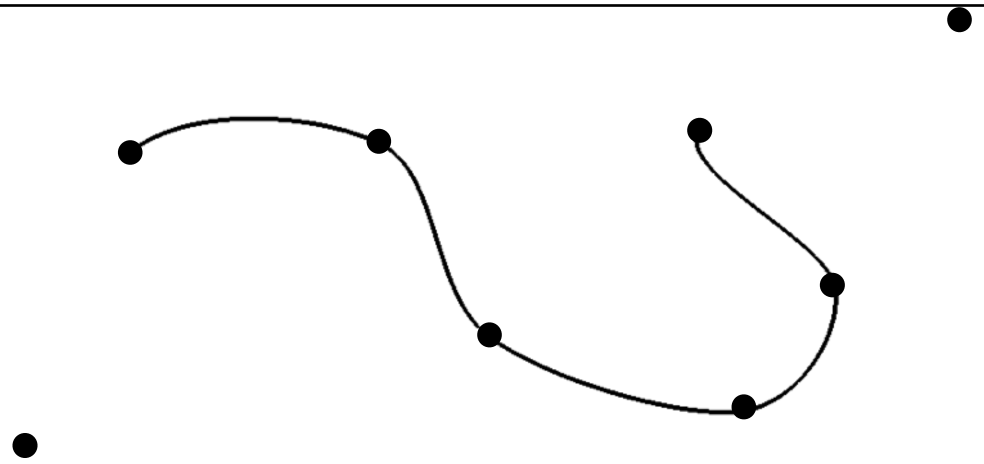
$$BF_0(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_1(u) = \ \tfrac{1}{2}u^3 - \ u^2 + \quad + \tfrac{2}{3}$$
$$BF_2(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_3(u) = \ \tfrac{1}{6}u^3$$

$$\begin{tikzcd}
BF_0'(0) = -\tfrac{1}{2} \arrow[dr] & BF_0'(1) = 0 \\
BF_1'(0) = 0 \arrow[dr] & BF_1'(1) = -\tfrac{1}{2} \\
BF_2'(0) = \tfrac{1}{2} \arrow[dr] & BF_2'(1) = 0 \\
BF_3'(0) = 0 & BF_3'(1) = \tfrac{1}{2}
\end{tikzcd}$$

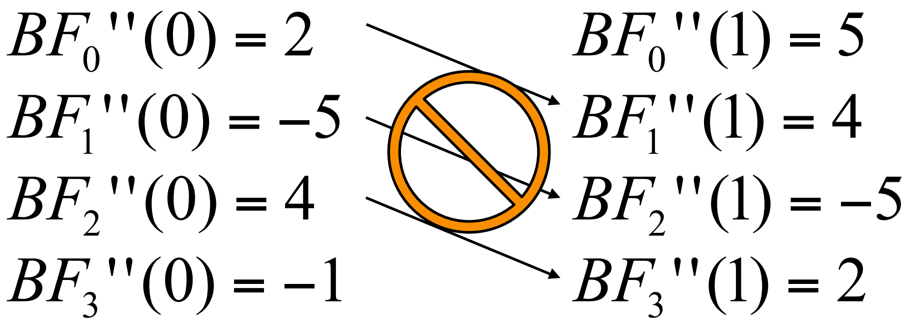
$$\boxed{P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}}$$

Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)

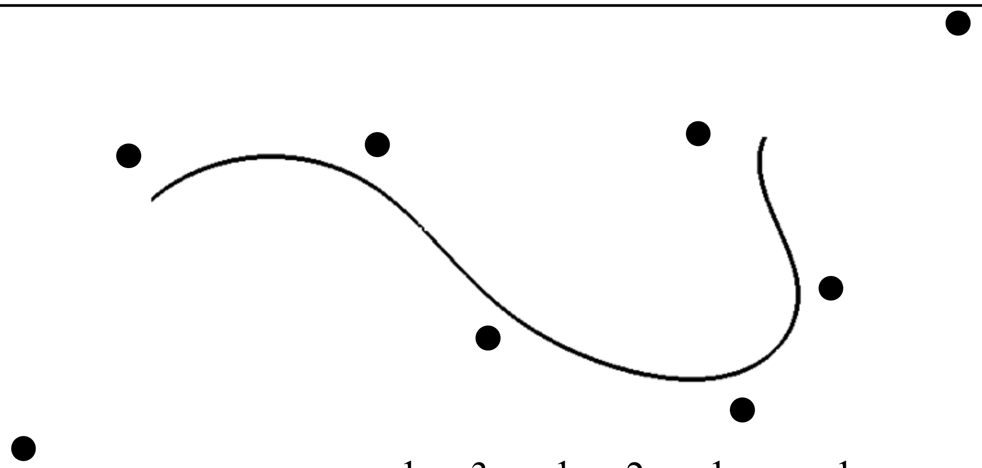


$$BF_0(u) = -\tfrac{1}{2}u^3 + \ \ u^2 - \tfrac{1}{2}u$$
$$BF_1(u) = \ \ \tfrac{3}{2}u^3 - \tfrac{5}{2}u^2 + \quad +1$$
$$BF_2(u) = -\tfrac{3}{2}u^3 + 2u^2 + \tfrac{1}{2}u$$
$$BF_3(u) = \ \ \tfrac{1}{2}u^3 - \tfrac{1}{2}u^2$$

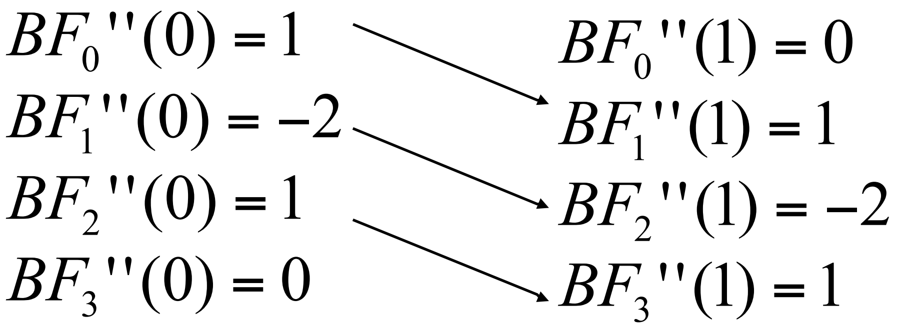


$$BF_0''(0) = 2 \qquad BF_0''(1) = 5$$
$$BF_1''(0) = -5 \qquad BF_1''(1) = 4$$
$$BF_2''(0) = 4 \qquad BF_2''(1) = -5$$
$$BF_3''(0) = -1 \qquad BF_3''(1) = 2$$

Cubic B-Splines



$$BF_0(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_1(u) = \ \ \tfrac{1}{2}u^3 - \ \ u^2 + \quad + \tfrac{2}{3}$$
$$BF_2(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_3(u) = \ \ \tfrac{1}{6}u^3$$



$$BF_0''(0) = 1 \qquad BF_0''(1) = 0$$
$$BF_1''(0) = -2 \qquad BF_1''(1) = 1$$
$$BF_2''(0) = 1 \qquad BF_2''(1) = -2$$
$$BF_3''(0) = 0 \qquad BF_3''(1) = 1$$

$$\boxed{P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}}$$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u)+ BF_1(u)+ BF_2(u)+ BF_3(u)=1$, for all $0 \leq u \leq 1$.
- Continuity:
  - $BF_0(1)=BF_3(0)=0$
  - $BF_1(1)=BF_0(0)$
  - $BF_2(1)=BF_1(0)$
  - $BF_3(1)=BF_2(0)$

Blending Functions

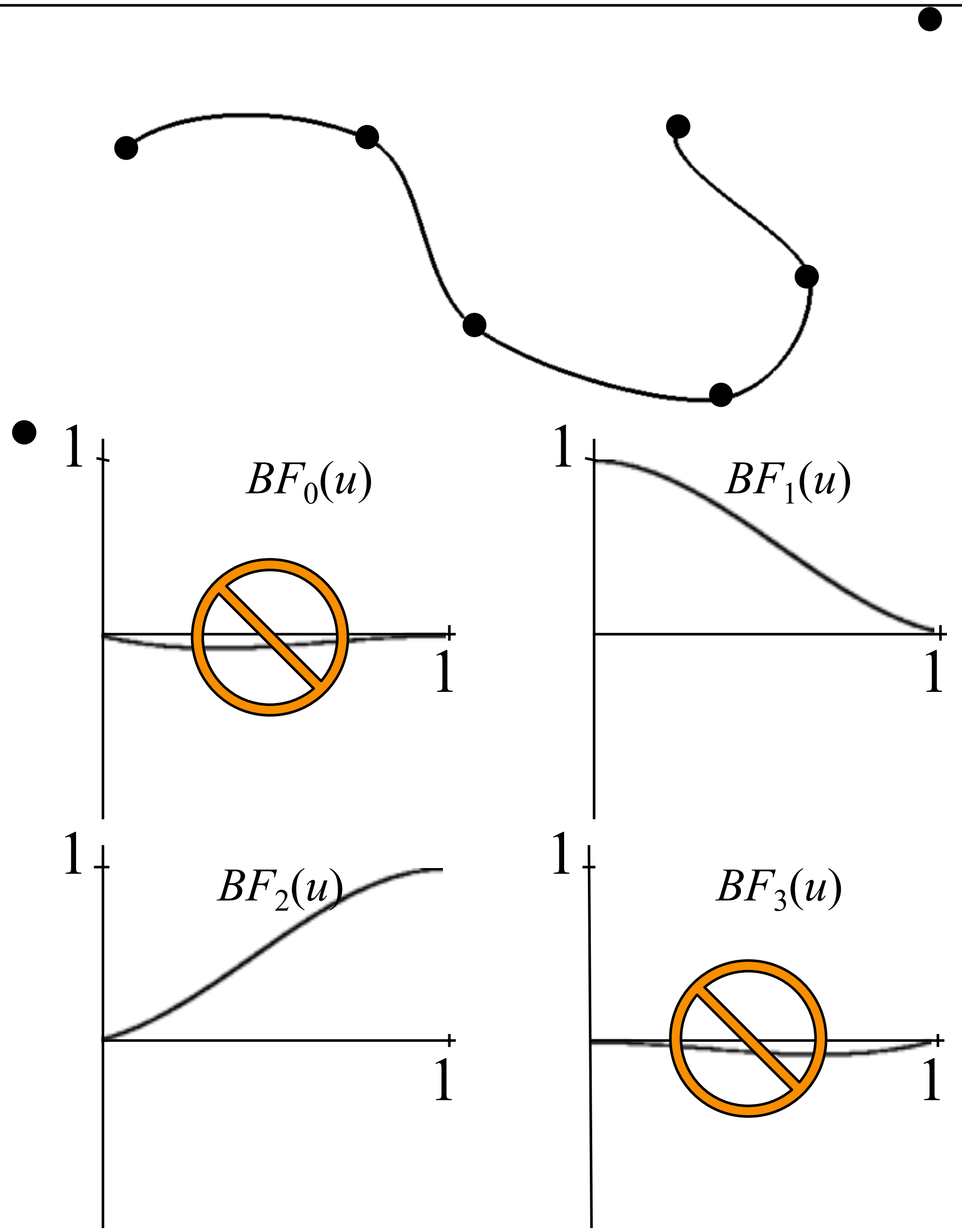
Properties:

- Translation Commutativity:
  - $BF_0(u) + BF_1(u) + BF_2(u) + BF_3(u) = 1$, for all $0 \leq u \leq 1$.
- Continuity:
  - $BF_0(1) = BF_3(0) = 0$
  - $BF_1(1) = BF_0(0)$
  - $BF_2(1) = BF_1(0)$
  - $BF_3(1) = BF_2(0)$
- Convex Hull Containment:
  - $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0$, for all $0 \leq u \leq 1$.

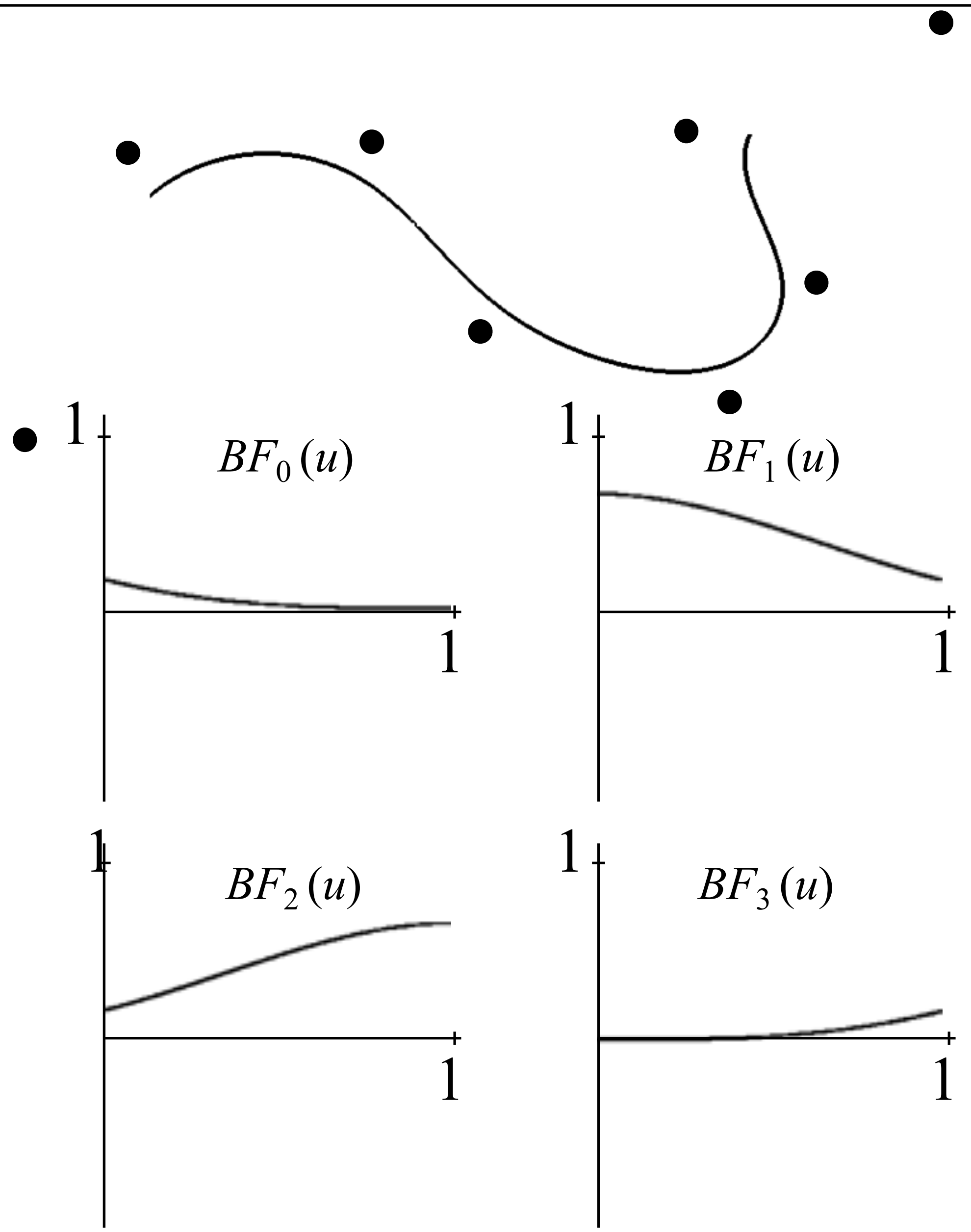
This is because a point is inside the convex hull of a collection of points if and only if it can be expressed as the weighted average of the points, where all the weights are non-negative.

Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)



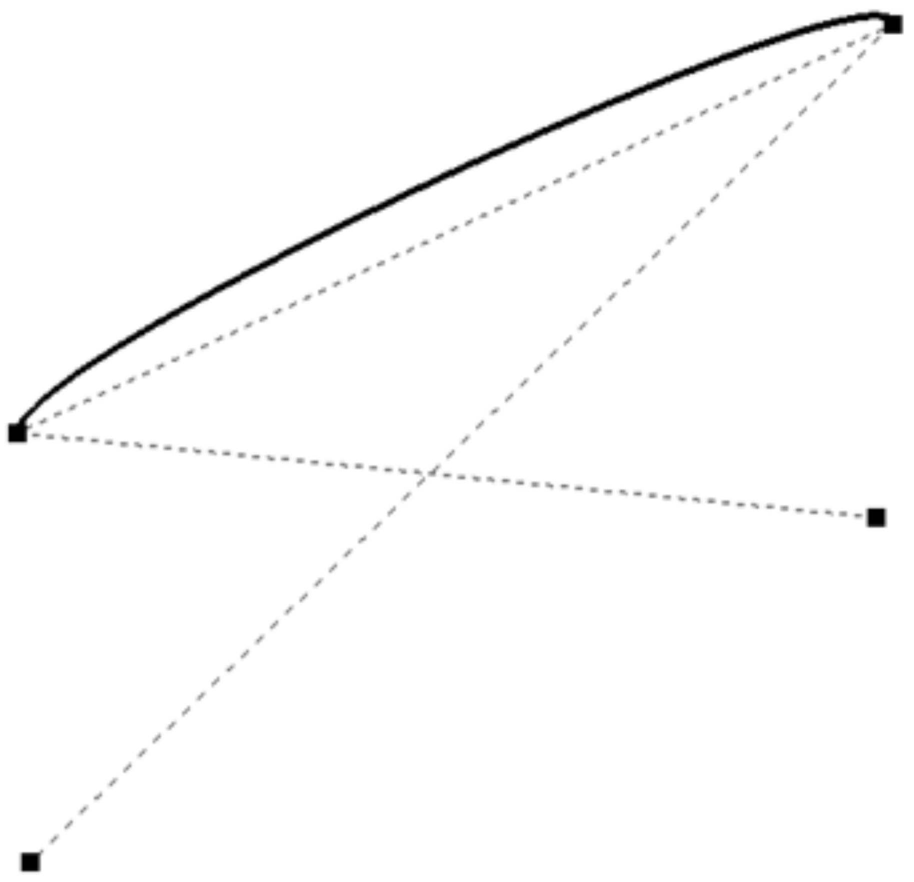
Cubic B-Splines



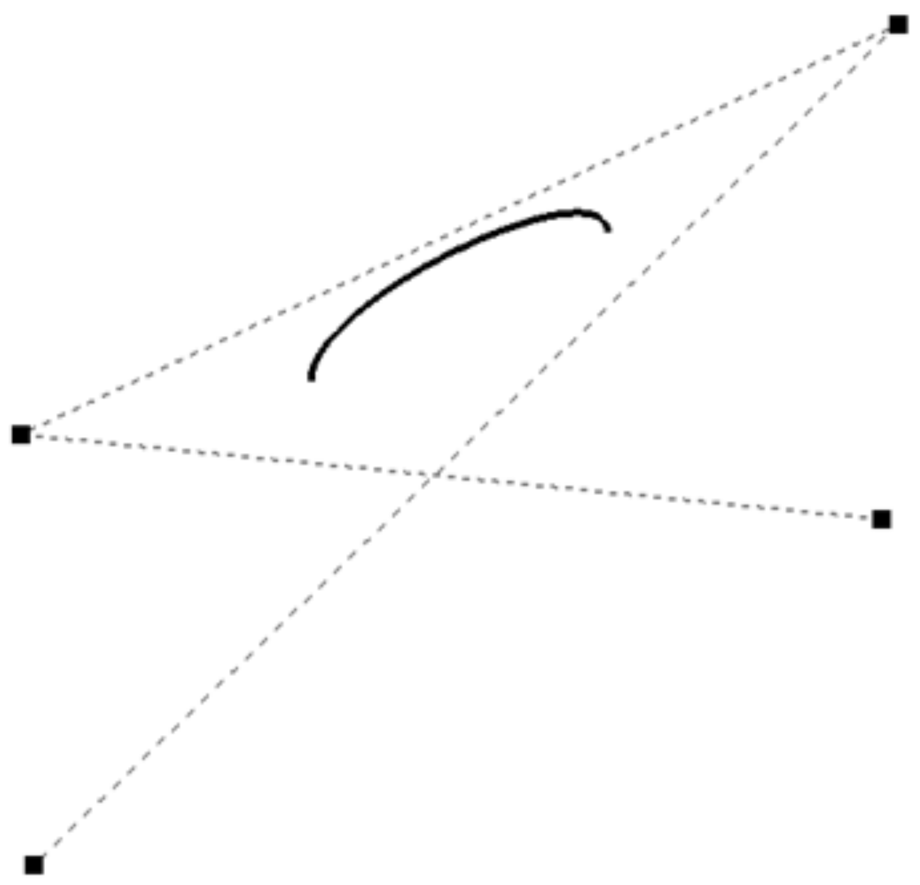
$$P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}$$

Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)



Cubic B-Splines



$$P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}$$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u)+ BF_1(u)+ BF_2(u)+ BF_3(u)=1$, for all $0 \leq u \leq 1$.
- Continuity:
  - $BF_0(1)=BF_3(0)=0$
  - $BF_1(1)=BF_0(0)$
  - $BF_2(1)=BF_1(0)$
  - $BF_3(1)=BF_2(0)$
- Convex Hull Containment:
  - $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0$, for all $0 \leq u \leq 1$.
- Interpolation:
  - $BF_0(0)=BF_2(0)=BF_3(0)=0$
  - $BF_0(1)=BF_1(1)=BF_3(1)=0$
  - $BF_1(0)=1$
  - $BF_2(1)=1$

Blending Functions

Properties:

- Translation Commutativity:
  - $BF_0(u)+ BF_1(u)+ BF_2(u)+ BF_3(u)=1$, for all $0 \leq u \leq 1$.

- Continuity:
  - $BF_0(1)=BF_3(0)=0$
  - $BF_1(1)=BF_0(0)$
  - $BF_2(1)=BF_1(0)$
  - $BF_3(1)=BF_2(0)$

- Convex Hull Containment:
  - $BF_0(u), BF_1(u), BF_2(u), BF_3(u) \geq 0$, for all $0 \leq u \leq 1$.

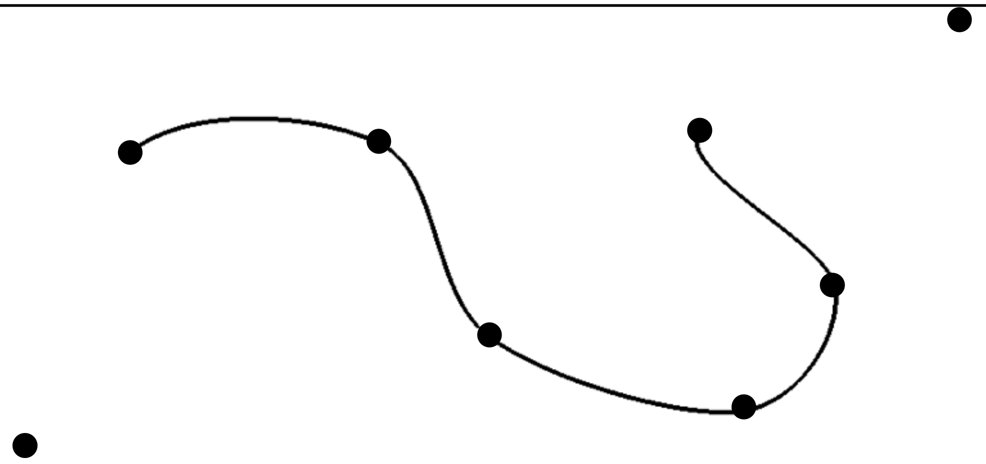
- Interpolation:
  - $BF_0(0)=BF_2(0)=BF_3(0)=0$
  - $BF_0(1)=BF_1(1)=BF_3(1)=0$
  - $BF_1(0)=1$
  - $BF_2(1)=1$

Because we want the spline segments to satisfy:

- $P_k(0)=p_{k+1}$
- $P_k(1)=p_{k+2}$

Comparison: Cardinal vs. Cubic B

Cardinal Splines (t=0)



$$BF_0(u) = -\tfrac{1}{2}u^3 + u^2 - \tfrac{1}{2}u$$
$$BF_1(u) = \tfrac{3}{2}u^3 - \tfrac{5}{2}u^2 + \quad + 1$$
$$BF_2(u) = -\tfrac{3}{2}u^3 + 2u^2 + \tfrac{1}{2}u$$
$$BF_3(u) = \tfrac{1}{2}u^3 - \tfrac{1}{2}u^2$$

$BF_0(0) = 0$

$\boxed{BF_1(0) = 1}$

$BF_2(0) = 0$

$BF_3(0) = 0$

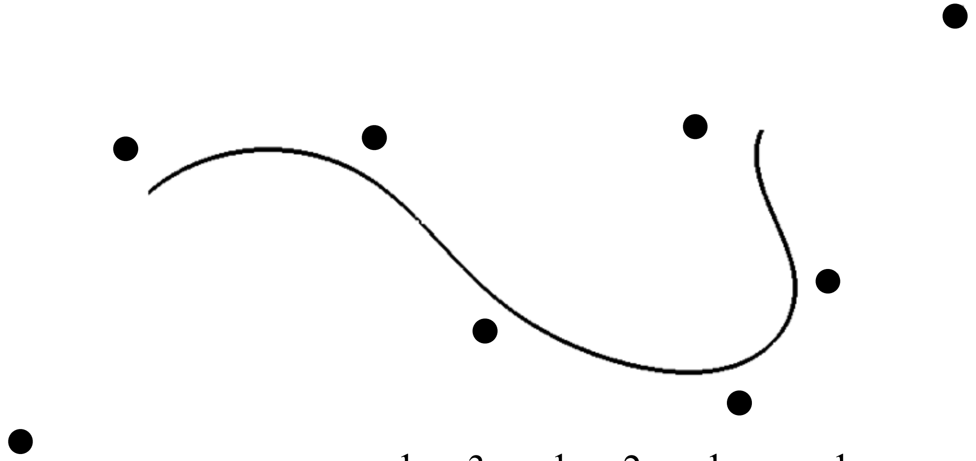
$BF_0(1) = 0$

$BF_1(1) = 0$

$\boxed{BF_2(1) = 1}$

$BF_3(1) = 0$

Cubic B-Splines



$$BF_0(u) = -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_1(u) = \tfrac{1}{2}u^3 - u^2 + \quad + \tfrac{2}{3}$$
$$BF_2(u) = -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6}$$
$$BF_3(u) = \tfrac{1}{6}u^3$$

$BF_0(0) = \frac{1}{6}$

$BF_1(0) = \frac{2}{3}$

$BF_2(0) = \frac{1}{6}$

$BF_3(0) = 0$

$BF_0(1) = 0$

$BF_1(1) = \frac{1}{6}$

$BF_2(1) = \frac{2}{3}$

$BF_3(1) = \frac{1}{6}$

$$\boxed{P_k(u) = BF_0(u)p_{k-1} + BF_1(u)p_k + BF_2(u)p_{k+1} + BF_3(u)p_{k+2}}$$

Summary

- A spline is a *piecewise* *polynomial function* whose derivatives satisfy some *continuity constraints* across curve junctions.

- Looked at specification for 3 splines:
    - Hermite
    - Cardinal
    - Uniform Cubic B-Spline

Hermite, Cardinal: Interpolating, cubic, C^1

Uniform Cubic B-Spline: Approximating, convex-hull containment, cubic, C^2