Animating Transformations

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CS445: Graphics

Acknowledgment: slides by Jason Lawrence, Misha Kazhdan, Allison Klein, Tom Funkhouser, Adam Finkelstein and David Dobkin

Overview

- Rotations and SVD
- Interpolating/Approximating Points
  - Vectors
  - Unit-Vectors
- Interpolating/Approximating Transformations
  - Matrices
  - Rotations
    - SVD Factorization
    - Euler Angles

Rotations

What are rotations?

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- A rotation *R* is a linear transformation that has determinant equal to one and preserves the angle between any two vectors:

$$\langle v, w \rangle = \langle R(v), R(w) \rangle$$

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Recall that the dot-product between two vectors can be expressed as a matrix multiplication:

$$\langle v, w \rangle = v^t w$$

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Since this is true for all v and w, this means that:

$$R^t R = \text{Identity} \quad \Longleftrightarrow \quad R^t = R^{-1}$$

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- A rotation R is a linear transformation that has determinant equal to one and whose transpose is equal to its inverse.

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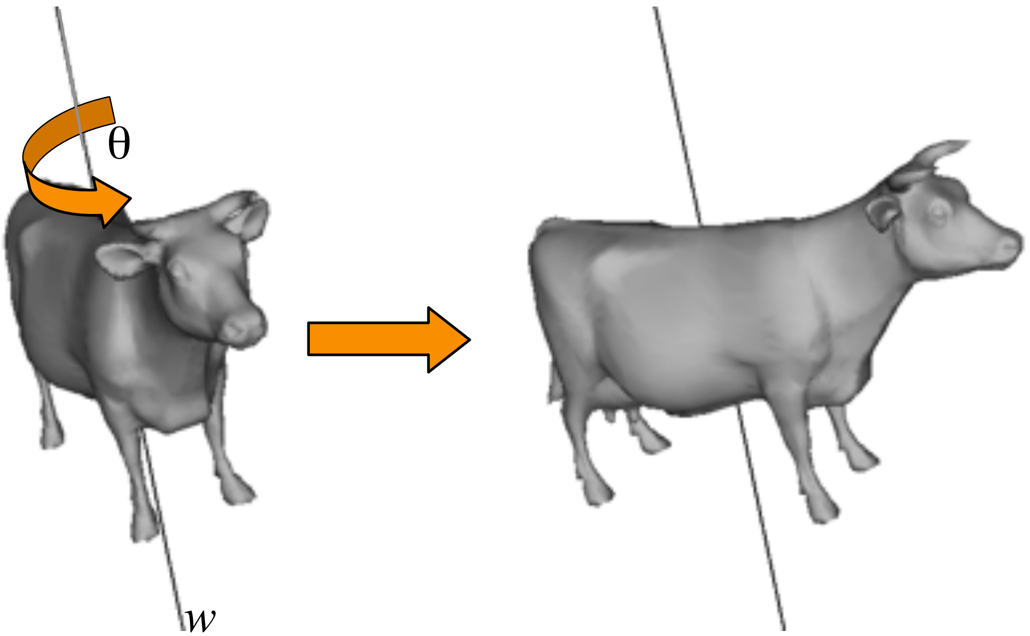
- A rotation in 3D can be specified by a 3x3 matrix.

Rotations

What are rotations?

- A rotation in 3D can also be specified by:
  - its axis of rotation w ($\|w\|=1$) and
  - its angle of rotation θ



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Properties:

- The rotation corresponding to (θ, w) is the same as the rotation corresponding to $(-\theta, -w)$.

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- Given two rotations corresponding to (θ_1, w) and (θ_2, w), the product of the rotations corresponds to $(\theta_1+\theta_2, w)$.
- Given a rotation corresponding (θ, w), the rotation raised to the power α corresponds to $(\alpha\theta, w)$.

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- Gi … ed to …

How do we define the product of rotations corresponding to (θ_1, w_1) and (θ_2, w_2)?

SVD

Any *m*x*n* matrix *M* can be expressed in terms of its Singular Value Decomposition as:

$$M = UDV^t$$

where:

- *U* is an *n*x*n* rotation matrix,
- *V* is an *m*x*m* rotation matrix, and
- *D* is an *m*x*n* diagonal matrix (i.e off-diagonals are all 0).

SVD

Applications:

- Compression
- Model Alignment
- Matrix Inversion
- Solving Over-Constrained Linear Equations

SVD

Matrix Inversion:

If we have an nxn invertible matrix M, we can use SVD to compute the inverse of M.

SVD

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If we have an *n*x*n* invertible matrix *M*, we can use SVD to compute the inverse of *M*.

Expressing *M* in terms of its SVD gives:

$$M = UDV^t$$

where:

- *U* is an *n*x*n* rotation matrix,
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We can express M^{-1} as:

$$M^{-1} = \left(UDV^t\right)^{-1}$$

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We can express M^{-1} as:

$$M^{-1} = \left(UDV^t\right)^{-1} = \left(V^{-1}\right)^t D^{-1} U^{-1}$$

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$$M^{-1} = \left(UDV^t\right)^{-1} = \left(V^{-1}\right)^t D^{-1} U^{-1}$$

$$= VD^{-1}U^t$$

Since:

- U is a rotation, $U^{-1}=U^t$.
- V is a rotation, $V^{-1}=V^t$.

SVD

Matrix Inversion:

If we have an *n*x*n* invertible matrix *M*, we can use SVD to compute the inverse of *M*.

We can express M^{-1} as:

$$M^{-1} = \left(UDV^t\right)^{-1} = \left(V^{-1}\right)^t D^{-1} U^{-1}$$

$$= VD^{-1}U^t$$

Since:

- D is a diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1/\lambda_n \end{pmatrix}$$

SVD

<u>Solving Over-Constrained Linear Equations</u>:

If we have *m* equations in *n* unknowns, with *m*>*n*, the problem is over-constrained and there is no general solution.

$$\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

SVD

Solving Over-Constrained Linear Equations:

If we have m equations in n unknowns, with $m \geq n$, the problem is over-constrained and there is no general solution.

$$\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_m \end{pmatrix}$$

However, using SVD, we can find the values of $\{x_1, \ldots, x_n\}$ that get us as close to $\{y_1, \ldots, y_m\}$ as possible.

SVD

Solving Over-Constrained Linear Equations:

If we express the matrix *A* in terms of its SVD:

$$A = UDV^t$$

then we can set the matrix A^* to be:

$$A^* = VD^*U^t$$

where D^* is the diagonal matrix with:

$$D^*_{ii} = \begin{cases} 1/D_{ii} & \text{if } D_{ii} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is called the pseudo-inverse of *A*.

That is, we invert *A* as much as possible.

SVD

Solving Over-Constrained Linear Equations:

If we set:

$$(x_1 \cdots x_n)^t = A^*(y_1 \cdots y_m)^t$$

this gives us the values of $\{x_1,\ldots,x_n\}$ that most nearly solve the initial equation:

$$A(x_1 \cdots x_n)^t = (y_1 \cdots y_m)^t$$

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Vectors

Given a collection of n control points $\{p_0,\ldots,p_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

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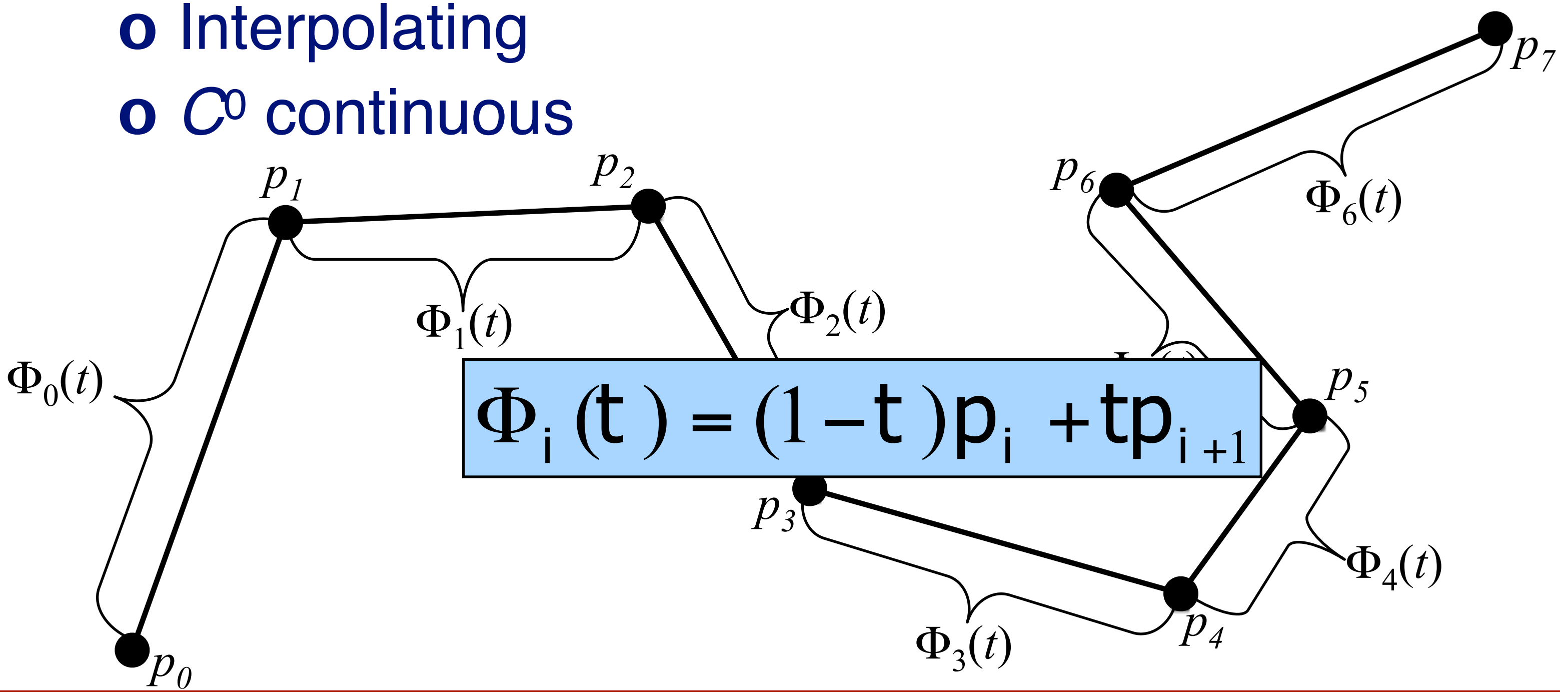
Linear Interpolation:

- Interpolating
- C^0 continuous



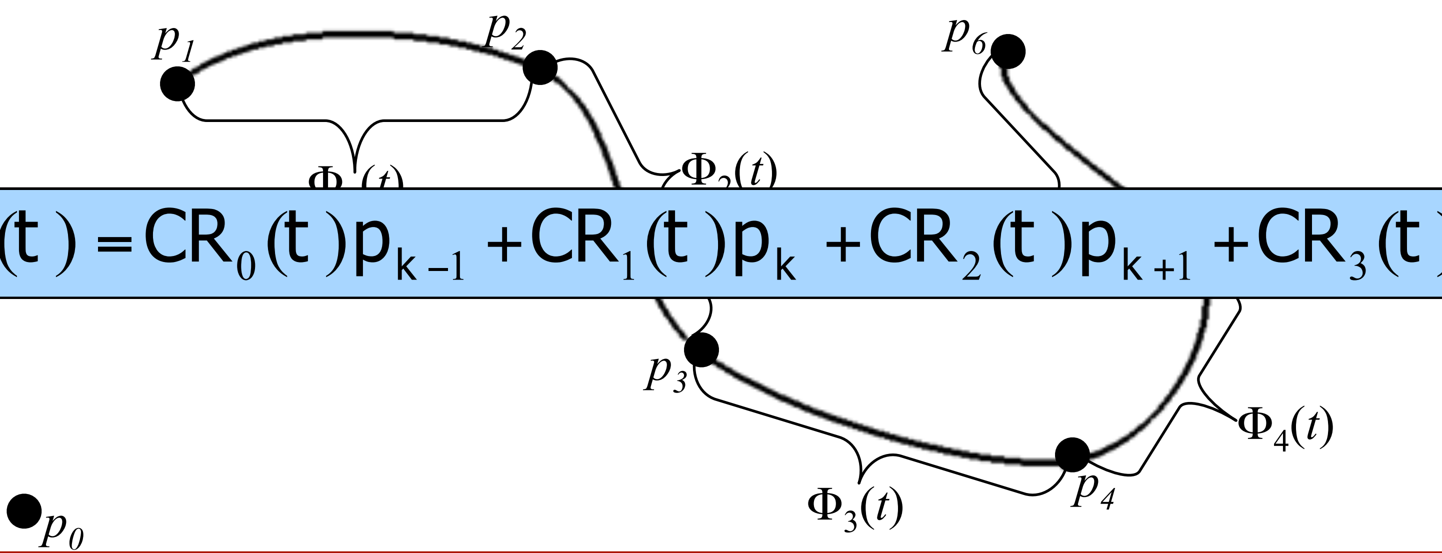
$$\Phi_i(t) = (1-t)p_i + tp_{i+1}$$

Vectors

Given a collection of *n* control points $\{p_0,\ldots,p_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

<u>Catmull-Rom Splines (Cardinal Splines with t=0)</u>:

- Interpolating
- C^1 continuous



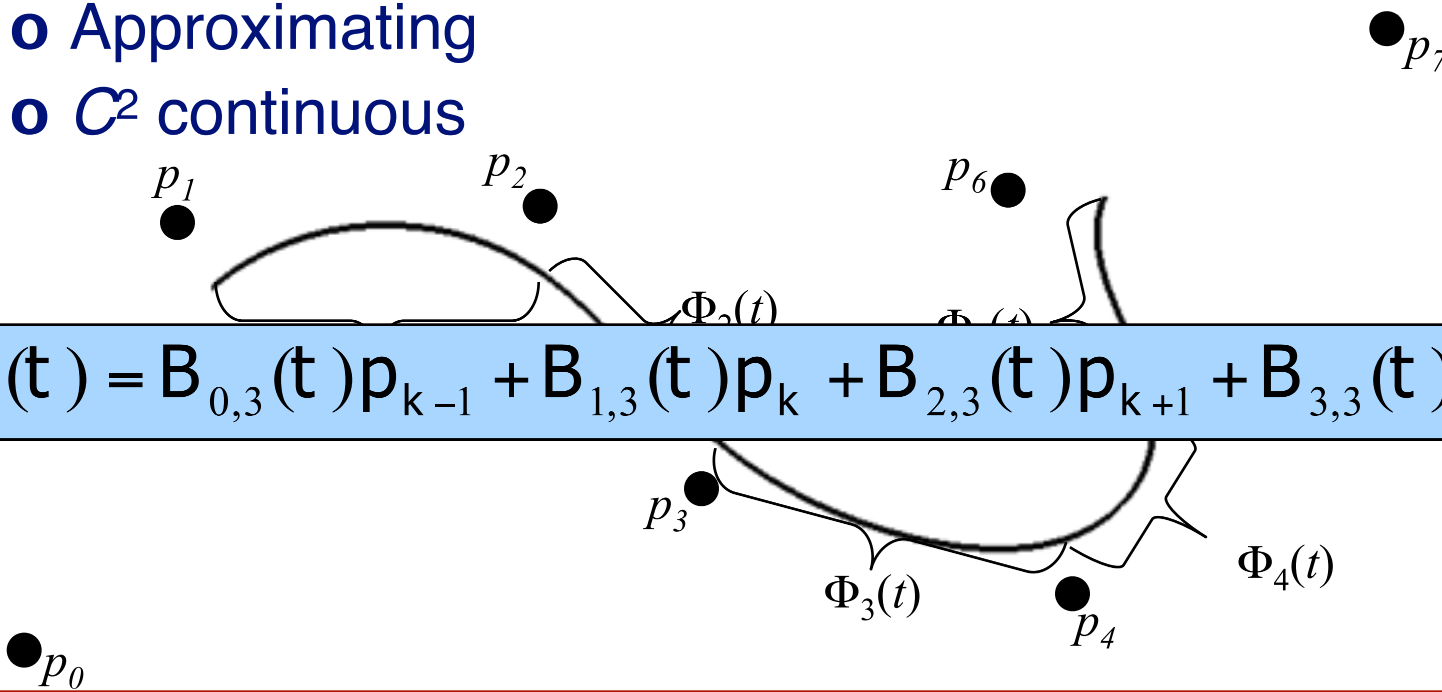
$$\Phi_i(t) = CR_0(t)p_{k-1} + CR_1(t)p_k + CR_2(t)p_{k+1} + CR_3(t)p_{k+2}$$

Vectors

Given a collection of *n* control points $\{p_0,\ldots,p_{n-1}\}$, define a curve $\Phi(t)$ that approximates/interpolates the points.

<u>Uniform Cubic B-Splines</u>:

- Approximating
- C^2 continuous



$$\Phi_i(t) = B_{0,3}(t)p_{k-1} + B_{1,3}(t)p_k + B_{2,3}(t)p_{k+1} + B_{3,3}(t)p_{k+2}$$

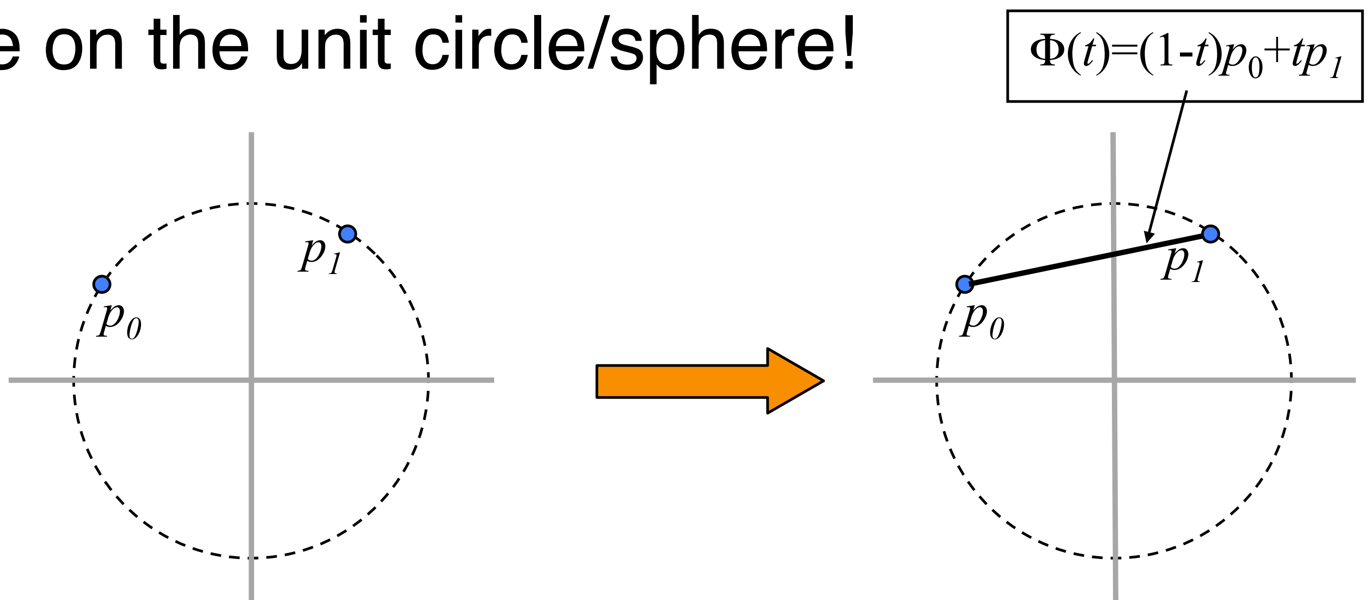
Unit-Vectors

What if we add the additional constraint that the points $\{p_0,\ldots,p_{n-1}\}$ and the curve $\Phi(t)$ have to lie on the unit circle/sphere ($\|p_i\|=1$, $\|\Phi(t)\|=1$)?

Unit-Vectors

What if we add the additional constraint that the points $\{p_0,\ldots,p_{n-1}\}$ and the curve $\Phi(t)$ have to lie on the unit circle/sphere ($\|p_i\|=1$, $\|\Phi(t)\|=1$)?

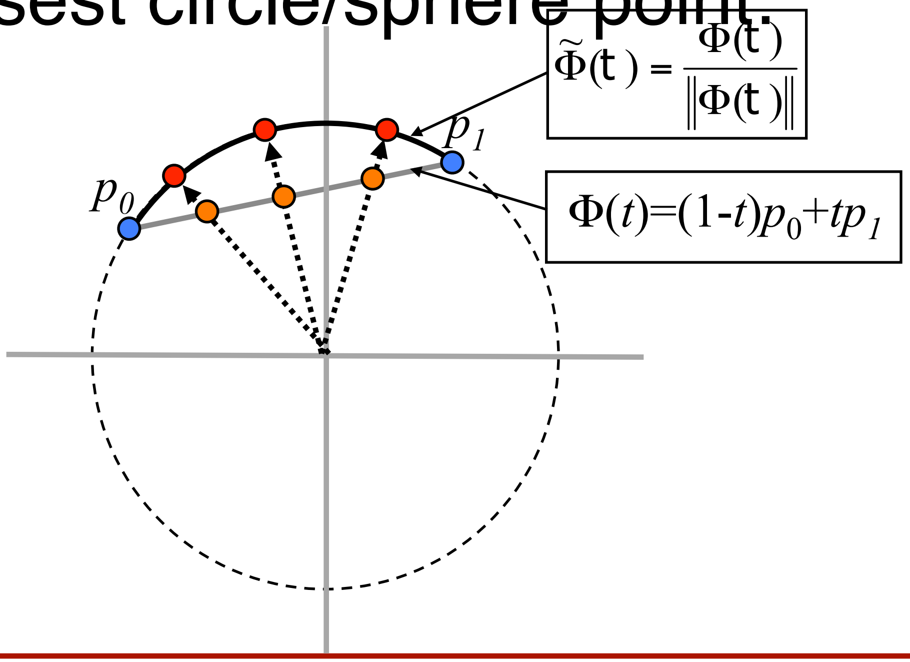
We can't interpolate/approximate the points as before, because the in-between points don't have to lie on the unit circle/sphere!



Unit-Vectors

What if we add the additional constraint that the points $\{p_0,\ldots,p_{n-1}\}$ and the curve $\Phi(t)$ have to lie on the unit circle/sphere ($\|p_i\|=1$, $\|\Phi(t)\|=1$)?

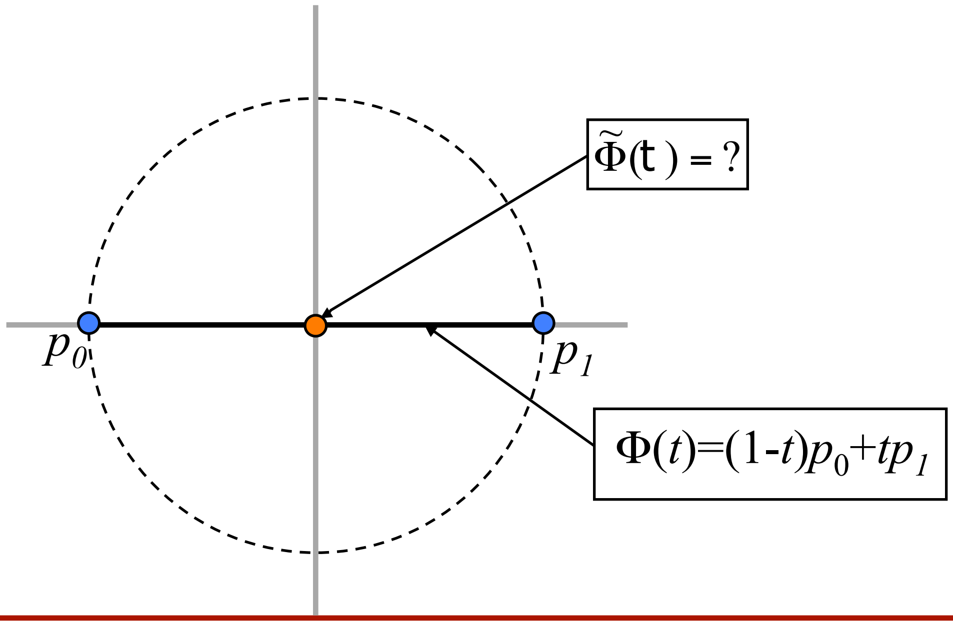
We can normalize the in-between points by sending them to the closest circle/sphere point:



Curve Normalization

<u>Limitations:</u>

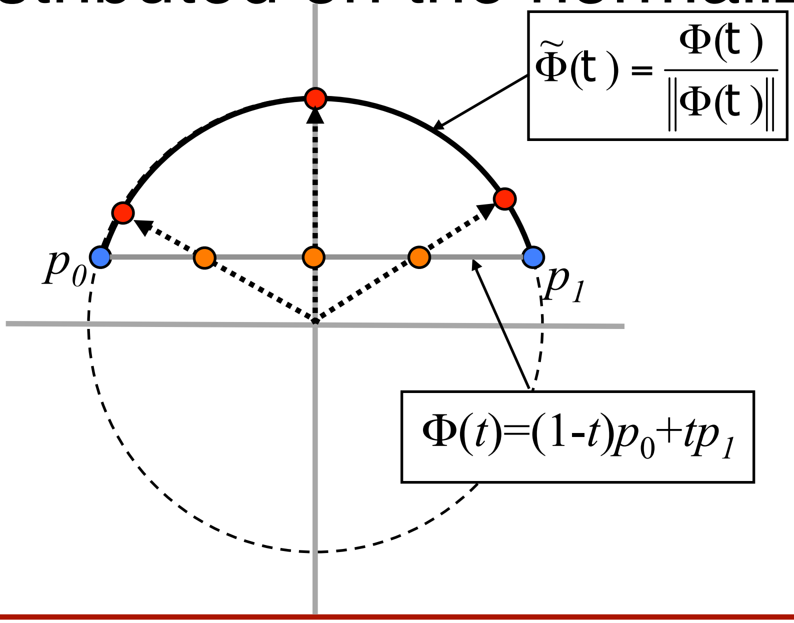
- The normalized curve is not always well defined.



Curve Normalization

Limitations:

- The normalized curve is not always well defined.
- Just because points are uniformly distributed on the original curve, does not mean that they will be uniformly distributed on the normalized one.



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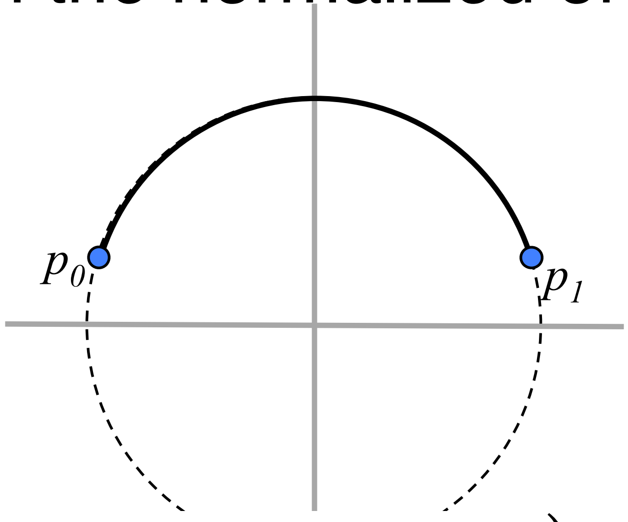
<u>SLERP</u>:

If we set:

- $p_0=(\cos\theta_0,\sin\theta_0)$
- $p_1=(\cos\theta_1,\sin\theta_1)$



We can set: $\Phi(t)=\left(\cos((1-t)\theta_0+t\theta_1),\sin((1-t)\theta_0+t\theta_1)\right)$

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<u>As with vectors:</u>

- Linear Interpolation:
$$\Phi_i(t) = (1-t)M_i + tM_{i+1}$$
- Catmull-Rom Interpolation:
$$\Phi_i(t) = CR_0(t)M_{k-1} + CR_1(t)M_k + CR_2(t)M_{k+1} + CR_3(t)M_{k+2}$$
- Uniform Cubic B-Spline Approximation:
$$\Phi_i(t) = B_{0,3}(t)M_{k-1} + B_{1,3}(t)M_k + B_{2,3}(t)M_{k+1} + B_{3,3}(t)M_{k+2}$$

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Rotations

- What if we add the additional constraint that the matrices $\{M_0,\ldots,M_{n-1}\}$ and the values of the curve $\Phi(t)$ have to be rotations?
- We can't interpolate/approximate the matrices as before, because the in-between matrices don't have to be rotations!
- We could try to normalize, by mapping every matrix $\Phi(t)$ to the nearest rotation.

Challenge

Given a matrix M, how do you find the rotation matrix R that is closest to M?

SVD Factorization

Given a matrix M, how do you find the rotation matrix R that is closest to M?

Singular Value Decomposition (SVD) allows us to express any M as a diagonal matrix, multiplied on the left and right by the rotations R_1 and R_2:

$$M = R_1 \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} R_2$$

SVD Factorization

Given a matrix *M*, how do you find the rotation matrix *R* that is closest to *M*?

Singular Value Decomposition (SVD) allows us to express any *M* as a diagonal matrix, multiplied on the left and right by the rotations R_1 and R_2:

To be fully correct, you need to ensure that the product of $\mathrm{sgn}(\lambda_i)$ is 1. If not, you need to flip the sign of the $\mathrm{sgn}(\lambda_i)$ where $|\lambda_i|$ is smallest.

The closest rotation *R* to *M* is then just the rotation:

$$R = R_1 \begin{pmatrix} \mathrm{sgn}(\lambda_1) & 0 & 0 \\ 0 & \mathrm{sgn}(\lambda_2) & 0 \\ 0 & 0 & \mathrm{sgn}(\lambda_3) \end{pmatrix} R_2$$

Euler Angles

Every rotation matrix *R* can be expressed as:

- some rotation about the *x*-axis, multiplied by
- some rotation about the *y*-axis, multiplied by
- some rotation about the *z*-axis:

$$R(\theta,\phi,\psi) = R_x(\theta)R_y(\phi)R_z(\psi)$$

The angles (θ,ϕ,ψ) are called the Euler angles.

Euler Angles

Instead of blending matrices and then normalizing using SVD, we can blend the Euler angles:

- For each M_k, compute the Euler angles $(\theta_k, \phi_k, \psi_k)$

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- Interpolate/Approximate the Euler angles:

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- For each M_k, compute the Euler angles (θ_k,ϕ_k,ψ_k)
- Interpolate/Approximate the Euler angles:
  - » Linear Interpolation:

$$\theta_k(t) = (1-t)\theta_k + t\theta_{k+1}$$

$$\phi_k(t) = (1-t)\phi_k + t\phi_{k+1}$$

$$\psi_k(t) = (1-t)\psi_k + t\psi_{k+1}$$

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  - » Catmull-Rom Interpolation:

$$\theta_k(t) = CR_0(t)\theta_{k-1} + CR_1(t)\theta_k + CR_2(t)\theta_{k+1} + CR_3(t)\theta_{k+2}$$

$$\phi_k(t) = CR_0(t)\phi_{k-1} + CR_1(t)\phi_k + CR_2(t)\phi_{k+1} + CR_3(t)\phi_{k+2}$$

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  - » Catmull-Rom Interpolation
  - » Uniform Cubic B-Spline Approximation:

$$\theta_k(t) = B_{0,3}(t)\theta_{k-1} + B_{1,3}(t)\theta_k + B_{2,3}(t)\theta_{k+1} + B_{3,3}(t)\theta_{k+2}$$

$$\phi_k(t) = B_{0,3}(t)\phi_{k-1} + B_{1,3}(t)\phi_k + B_{2,3}(t)\phi_{k+1} + B_{3,3}(t)\phi_{k+2}$$

$$\psi_k(t) = B_{0,3}(t)\psi_{k-1} + B_{1,3}(t)\psi_k + B_{2,3}(t)\psi_{k+1} + B_{3,3}(t)\psi_{k+2}$$

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- Interpolate/Approximate the Euler angles:
  - » Linear Interpolation
  - » Catmull-Rom Interpolation
  - » Uniform Cubic B-Spline Approximation
- Set the value of the in-between matrix to:

$$\Phi_i(t) = R_x(\theta_i(t))R_y(\phi_i(t))R_z(\psi_i(t))$$